Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2008

Solutions to Assignment #4
A quadratic equation

1. Find all the matrices $\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfying the equation $\mathbf{X}^2 - 2\mathbf{X} + \mathbf{I}_2 = \mathbf{0}$. *[10]*

Note: $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ satisfies the equation, but it's not the only one that does. For example, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ does so too.

Solution. Observe that $\mathbf{X}^2 - 2\mathbf{X} + \mathbf{I}_2 = (\mathbf{X} - \mathbf{I}_2)^2$. If $\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\begin{aligned}(\mathbf{X} - \mathbf{I}_2)^2 &= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^2 = \begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix}^2 \\ &= \begin{bmatrix} (a-1)^2 + bc & b(a+d-2) \\ c(a+d-2) & (d-1)^2 + bc \end{bmatrix} .\end{aligned}$$

Finding the solutions to the equation $(\mathbf{X} - \mathbf{I}_2)^2 = \mathbf{X}^2 - 2\mathbf{X} + \mathbf{I}_2 = \mathbf{0}$ therefore boils down to finding the solutions of the following system of (non-linear!) equations:

$$\begin{aligned}(a-1)^2 + bc &= 0 \\ (d-1)^2 + bc &= 0 \\ b(a+d-2) &= 0 \\ c(a+d-2) &= 0\end{aligned}$$

To get a handle on finding all the solutions to this system of equations, we divide it up into manageable cases and then conquer each one.

Case I – $bc = 0$: Note that $bc = 0$ exactly when $b = 0$, $c = 0$, or both. In this case, the first two equations in the system imply that $(a-1)^2 = (d-1)^2 = 0$, so $a - 1 = d - 1 = 0$, *i.e.* $a = d = 1$. Since it follows from this that $a + d - 2 = 0$, the last two equations in the system will be satisfied even if one of b or c is 0.

This gives us two families of infinitely many solutions each:

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$$

$$b \in \mathbb{R} \qquad c \in \mathbb{R}$$

Case II – $bc \neq 0$: First, note that $bc \neq 0$ exactly when $b \neq 0$ and $c \neq 0$. It follows from the last two equations in the system that $a + d - 2 = 0$.

Second, in this case the first two equations tell us less than when $bc = 0$, as all we can conclude is that $(a-1)^2 = (d-1)^2$. From this it follows that either $a - 1 = d - 1$ or $a - 1 = -(d-1) = 1 - d$, *i.e.* $a = d$ or $a + d = 2$. We analyze these subcases separately:

Subcase II.i – $a = d$: We already know that in Case II, $a + d - 2 = 0$. If $a = d$ as well, we must have $a = d = 1$. However, the first two equations in the system then imply that $(1-1)^2 + bc = 0 = (1-1)^2 + bc$, so $bc = 0$. This is a contradiction to our being in Case II at all, so this subcase does not lead to any new solutions.

Subcase II.ii – $a + d = 2$: In this subcase, all we know about a and d is what $a + d - 2 = 0$ already told us. We can solve for d in terms of a, $d = 2 - a$, or *vice versa*, $a = 2 - d$, but that's as far as that goes. Note that the argument showing that Subcase II.i leads to a contradiction shows us that here we need to have $a \neq 1$ to avoid contradicting $bc \neq 0$. (Similarly, $d \neq 1$.)

Given $a \neq 1$ and $b \neq 0$, we can use the first equation in the system to solve for c: $c = -\frac{(a-1)^2}{b}$. Note that this means that $c \neq 0$ and that c and b must have opposite signs. This gives us one more infinite family of solutions:

$$\begin{bmatrix} a & b \\ -\frac{(a-1)^2}{b} & 2-a \end{bmatrix}$$

$$a \neq 1 \text{ and } b \neq 0$$

Since we have exhausted all the logical possibilities between our various (sub)cases, we have found all the possible solutions for $\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfying $\mathbf{X}^2 - 2\mathbf{X} + \mathbf{I}_2 = \mathbf{0}$. ■