Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2008

Solutions to Assignment #3

Linear optimization

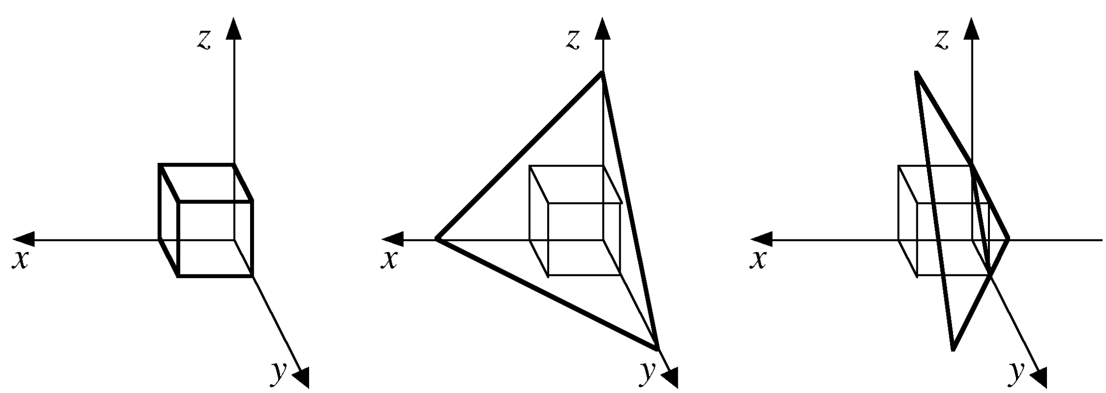
In this assignment we will deal with the solid whose faces are (parts of) the planes given by the equations $x = 0$, $y = 0$, $z = 0$, $x = 10$, $y = 10$, $z = 10$, $x + y + z = 25$, and $2x - y - z = -10$. Another way to look at this solid is as the set of points with coordinates (x, y, z) which satisfy *all* of the following seven inequalities: $x \geq 0$, $y \geq 0$, $z \geq 0$, $x \leq 10$, $y \leq 10$, $z \leq 10$, $x + y + z \leq 25$, and $2x - y - z \geq -10$.

1. Find the coordinates of all of the vertices of this solid and make as accurate a sketch as you can of it. *[6]*

Solution. The vertices of the solid are those points where three or more of the defining planes meet and which satisfy all of the corresponding inequalities.

A first cut at the solid can be had quickly by considering the first six planes, $x = 0$, $y = 0$, $z = 0$, $x = 10$, $y = 10$, and $z = 10$, and the corresponding inequalities. It easy to see that these define a cube with the following vertices: $(0, 0, 0)$, $(10, 0, 0)$, $(0, 10, 0)$, $(0, 0, 10)$, $(10, 10, 0)$, $(10, 0, 10)$, $(0, 10, 10)$, and $(10, 10, 10)$. Checking these against the other two inequalities, $x + y + z \leq 25$ and $2x - y - z \geq -10$, reveals that $(10, 10, 10)$ fails to satisfy $x + y + z \leq 25$ and that $(0, 10, 10)$ fails to satisfy $2x - y - z \geq -10$, so these two points do not end up being vertices of our solid. The other vertices of the cube do satisfy both of these inequalities and so end up being vertices of the solid.

To get a better idea of how the planes $x + y + z = 25$ and $2x - y - z = -10$ cut off parts of the cube to make our solid, we sketch them and compare the sketches to a sketch of the cube on the same scale. Note that $x + y + z = 25$ has intercepts $(25, 0, 0)$, $(0, 25, 0)$, and $(0, 0, 25)$, while $2x - y - z = -10$ has intercepts $(-5, 0, 0)$, $(0, 10, 0,)$, and $(0, 0, 10)$, the last two of which are vertices of the cube that survive to be vertices of the solid.

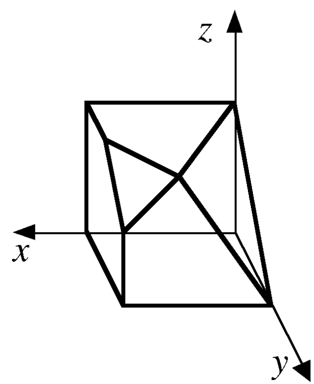


Note the non-standard orientation of the axes; this was done to make it easier to see what the plane $2x - y - z = -10$ is doing.

It is pretty clear from the second these pictures that if $x + y + z = 25$ cuts off the corner $(10, 10, 10)$ of the cube, it will cut through the edges joining this corner to the adjacent vertices, namely $(10, 10, 0)$, $(10, 0, 10)$, and $(0, 10, 10)$. These edges are given by the intersections of the planes $x = 10$, $y = 10$, and $z = 10$, taken two at a times: plugging in 10 for two of the three variables in $x+y+z = 25$ and solving for the third, gives us the points $(5, 10, 10)$, $(10, 5, 10)$, and $(10, 10, 5)$, all of which satisfy all of the given inequalities.

It should also be pretty clear from the third of the pictures that $2x - y - z = -10$ cuts through the edge of the cube joining the vertices $(10, 10, 10)$ and $(0, 10, 10)$. This edge is given by the intersection of the planes $y = 10$ and $z = 10$; plugging $y = z = 10$ into $2x - y - z = -10$ and solving for x gives the point $(5, 10, 10)$.

Thus the complete list of vertices of our solid is $(0, 0, 0)$, $(10, 0, 0)$, $(0, 10, 0)$, $(0, 0, 10)$, $(10, 10, 0)$, $(10, 0, 10)$, $(5, 10, 10)$, $(10, 5, 10)$, and $(10, 10, 5)$. Here's the picture:



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2. Find the maximum value of the function $f(x, y, z) = 2x + 2y + z$ on this solid and determine at which point(s) of the solid this maximum occurs. *[4]*

Solution. As was noted in the tutorials, the maximum of a linear function on a polyhedron will occur at a vertex of the polyhedron. We will therefore compare the values of $f(x, y, z)$ on the vertices of our solid:

Vertex (x, y, z)	$f(x, y, z)$
$(0, 0, 0)$	0
$(10, 0, 0)$	20
$(0, 10, 0)$	20
$(0, 0, 10)$	10
$(10, 10, 0)$	40
$(10, 0, 10)$	30
$(10, 10, 5)$	45
$(10, 5, 10)$	40
$(5, 10, 10)$	40

Thus the maximum value of $f(x, y, z)$ on a vertex of the solid is 45 and occurs at $(10, 10, 5)$.

How can we be sure that $f(10, 10, 5) = 45$ is the maximum value of $f(x, y, z) = 2x + 2y + z$ on the solid in question? Observe that

$$\begin{aligned} f(x, y, z) &= 2x + 2y + z \\ &= x + y + (x + y + z) \\ &\leq 10 + 10 + 25 \\ &\quad \text{Since } x \leq 10,\ y \leq 10, \text{ and } x + y + z \leq 25 \text{ for all points on the solid.} \\ &= 45 \end{aligned}$$

Hence $f(10, 10, 5) = 45$ is the maximum va;ue of $f(x, y, z) = 2x + 2y + z$ that can be achieved on the solid. ■