Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2008

SOLUTIONS TO THE FINAL EXAMINATION
Friday, 19 December, 2008

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Show all your work. *If in doubt about something,* **ask!**

Aids: Calculator; annotated *Formula for Success* or $8.5'' \times 11''$ aid sheet; one brain.

Part I. Do all of **1–5**.

1. Consider the plane in $\mathbb{R}^3$ given by the equation $x + y + 2z = 8$, and the line in $\mathbb{R}^3$ given by the parametric equations $x = 4 + t$, $y = 4 - t$, and $z = 0$.

a. Sketch the parts of this plane and line that lie in the first octant. *[4]*

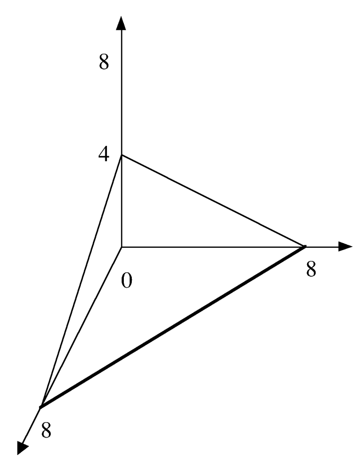
b. Find the angle between the normal vector of the plane and the direction vector of the line. *[3]*

c. Show that the line is contained in the plane. *[3]*

Solution to a. To sketch the part of the plane in the first octant, we first find its intercepts. The intercepts of the plane are obtained by setting two of x, y, and z to 0 in $x + y + 2z = 8$ and the solving for the third. This gives the points $(8, 0, 0)$, $(0, 8, 0)$, and $(0, 0, 4)$. Connecting these up gives us the part of the plane for which x, y, and z are all ≥ 0, *i.e.* the part of the plane in the first octant.

To sketch the part of the line in the first octant, we need is to find which points on the line have all of its coordinates ≥ 0. Note that $x = 4 + t \geq 0$ exactly when $t \geq -4$, $y = 4 - t \geq 0$ exactly when $t \leq 4$, and $z = 0 \geq 0$ for all t. It follows that the points on the line in the first octant are those given by $-4 \leq t \leq 4$. The point corresponding to $t = -4$ is $(0, 8, 0)$, and the point corresponding to $t = 4$ is $(8, 0, 0)$. Plot these two points and connect them up ...

Here's the sketch:



Note that the axes, which are not labelled, are in standard position. ■

Solution to b. The normal vector to the plane and the direction vector of the line are $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, respectively. Recall that the angle θ between them satisfies $\cos(\theta) = \dfrac{\mathbf{n} \cdot \mathbf{d}}{\|\mathbf{n}\|\|\mathbf{d}\|}$. Since $\mathbf{n} \cdot \mathbf{d} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1 \cdot 1 + 1 \cdot (-1) + 2 \cdot 0 = 1 - 1 + 0 = 0$, $\cos(\theta) = 0$, which implies that $\theta = 90°$, *i.e.* $\mathbf{n}$ and $\mathbf{d}$ are orthogonal to each other. ■

Solution to c. As should be obvious from part **a**, two points of the line are in the plane, namely $(8, 0, 0)$ and $(0, 8, 0)$. It follows that the entire line must be in the plane.

More directly, plugging the parametric description of the line, $x = 4 + t$, $y = 4 - t$, and $z = 0$, into the left-hand side of the equation of the plane, $x + y + 2z = 8$, gives

$$(4 + t) + (4 - t) + 2 \cdot 0 = 4 + t + 4 - t + 0 = 8,$$

which equals the right-hand side of the equation of the plane. It follows that every point of the line is a point of the plane, *i.e.* the line is contained in the plane. ■

2. Consider the following system of linear equations.

$$\begin{array}{rcrcrcrcr} w & + & & & 2y & + & 4z & = & -8 \\ & & x & - & 3y & - & z & = & 6 \\ 3w & + & 4x & - & 6y & + & 8z & = & 0 \\ & - & x & + & 3y & + & 4z & = & -12 \end{array}$$

Use Gaussian elimination to find all the solutions, if any, of this system. Without computing it, what must the determinant of the matrix of coefficients be? *[10]*

Solution. We set up the corresponding augmented matrix and throw the Gauss-Jordan algorithm at it:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right] \underset{R_3 - 3R_1}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 4 & -12 & -4 & 24 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right]$$

$$\underset{\substack{R_3 - 4R_2 \\ R_4 + R_2}}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -6 \end{array}\right] \underset{\frac{1}{3}R_4}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\underset{R_3 \leftrightarrow R_4}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{\substack{R_1 - 4R_3 \\ R_2 + R_3}}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Having taken Gauss-Jordan reduction as far as we can, the last matrix corresponds to the system of equations $w + 2y = 0$, $x - 3y = 4$, and $z = 0$, *i.e.* $w = -2y$, $x = 4 + 3y$, and $z = 0$. If we set y equal to the parameter t, we can write all the possible solutions in vector-parametric form:

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{where } t \in \mathbb{R}$$

Since the 4×4 matrix of coefficients can be row-reduced to a matrix with a row of zeroes, it must have determinant 0. ■

3. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$.

a. Find the inverse of $\mathbf{A}$, if it exists. *[10]*

b. Use your work in **a** to compute the determinant of $\mathbf{A}$. *[5]*

Solution to a. We set up the "super-augmented" matrix $[\,\mathbf{A} \mid \mathbf{I}\,]$ and try to reduce it to $\left[\,\mathbf{I} \mid \mathbf{A}^{-1}\,\right]$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ \Longrightarrow \\ R_3 - 5R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ (-1)R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_3 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right]$$

Thus $\mathbf{A}^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$. ■

Solution to b. The only row operation that would change the determinant of $\mathbf{A}$ in the (left sides only of the) reduction above is the next-to-last step, $(-1)R_3$. It follows that $|\mathbf{A}|(-1) = |\mathbf{I}| = 1$, so $|\mathbf{A}| = -1$. ■

4. Find all the eigenvalues and eigenvectors of $\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$. *[15]*

Solution. To find the eigenvalues we solve the equation $|\mathbf{C} - \lambda\mathbf{I}| = 0$ for λ.

$$\begin{aligned} |\mathbf{C} - \lambda\mathbf{I}| &= \left| \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 2-\lambda & 1 \\ 3 & 0-\lambda \end{vmatrix} \\ &= (2-\lambda)(-\lambda) - 1 \cdot 3 = \lambda^2 - 2\lambda - 3 \end{aligned}$$

We thus need to solve $\lambda^2 - 2\lambda - 3 = 0$ for λ. Using the quadratic equation, we get:

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = 1 \pm 2$$

Thus $\lambda = 1 + 2 = 3$ and $\lambda = 1 - 2 = -1$ are the eigenvalues of $\mathbf{C}$.

To find all the eigenvectors of $\mathbf{C}$ we need to solve the homogenous equations $(\mathbf{C} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for each eigenvalue.

When $\lambda = 3$,

$$\mathbf{C} - \lambda\mathbf{I} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}.$$

We row-reduce the augmented matrix for the corresponding homogeneous system of linear equations:

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 3 & -3 & 0 \end{array}\right] \underset{R_2 + 3R_1}{\Longrightarrow} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \overset{(-1)R_1}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The corresponding linear equation is $x - y = 0$, *i.e.* $x = y$. If we set y equal to the parameter t, the set of possible solutions, *i.e.* the eigenvectors of $\mathbf{C}$ corresponding to the eigenvalue $\lambda = 3$, are $\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where $t \in \mathbb{R}$.

Similarly, when $\lambda = -1$,

$$\mathbf{C} - \lambda\mathbf{I} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}.$$

We row-reduce the augmented matrix for the corresponding homogeneous system of linear equations:

$$\left[\begin{array}{cc|c} 3 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right] \underset{R_2 - R_1}{\Longrightarrow} \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \overset{\frac{1}{3}R_1}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The corresponding linear equation is $x + \frac{y}{3} = 0$, *i.e.* $x = -\frac{y}{3}$. If we set y equal to the parameter s, the set of possible solutions, *i.e.* the eigenvectors of $\mathbf{C}$ corresponding to the eigenvalue $\lambda = 3$, are $\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$, where $s \in \mathbb{R}$. ■

5. Let $\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

a. Find a basis for the column space of $\mathbf{M}$. *[10]*
b. Determine the rank and nullity of $\mathbf{M}$. *[5]*

Solution to a. We will row-reduce $\mathbf{M}^T = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 1 \end{bmatrix}$ as for Gaussian, rather than Gauss-Jordan, elimination; the surviving non-zero rows, transposed back into columns, will be the basis we desire.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 1 \end{bmatrix} \underset{R_4 + R_1}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\underset{R_2 \leftrightarrow R_3}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \underset{R_4 - R_2}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the column space of $\mathbf{M}$. ■

Solution to b. The rank of $\mathbf{M}$ is equal to the dimension of its column space, which must be 3 because there are three vectors in the basis obtained in part **a**. Since $\mathbf{M}$ is a 4×5 matrix, its nullity is 4 minus its rank, *i.e.* $4 - 3 = 1$. ■

Part II. Do any *three* of **6–11**.

6. Suppose T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad\text{and}\quad T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Find the matrix $\mathbf{D}$ such that for all $\mathbf{x}\in\mathbb{R}^3$, $T(\mathbf{x})=\mathbf{D}\mathbf{x}$. *[10]*

Solution. We need to find a matrix $\mathbf{D}$ such that

$$\mathbf{D}\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \mathbf{D}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix},\ \text{and}\quad \mathbf{D}\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

This amounts to finding a matrix $\mathbf{D}$ such that

$$\mathbf{D}\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix};$$

that is,

$$\mathbf{D}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}^{-1}=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}^{-1},$$

assuming that the inverse in question exists. We attempt, therefore, to compute said inverse:

$$\left[\begin{array}{ccc|ccc}0&1&1&1&0&0\\1&0&1&0&1&0\\1&1&0&0&0&1\end{array}\right]\ \underset{}{\overset{R_1\leftrightarrow R_2}{\Longrightarrow}}\ \left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&1&1&0&0\\1&1&0&0&0&1\end{array}\right]$$

$$\underset{R_3-R_1}{\Longrightarrow}\ \left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&1&1&0&0\\0&1&-1&0&-1&1\end{array}\right]\ \underset{R_3-R_2}{\Longrightarrow}\ \left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&1&1&0&0\\0&0&-2&-1&-1&1\end{array}\right]$$

$$\underset{\left(-\frac{1}{2}\right)R_3}{\Longrightarrow}\ \left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&1&1&0&0\\0&0&1&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\end{array}\right]\ \underset{\Longrightarrow}{\overset{R_1-R_3}{R_2-R_3}}\ \left[\begin{array}{ccc|ccc}1&0&0&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\0&1&0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\0&0&0&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\end{array}\right]$$

Thus the inverse matrix exists, and so

$$\mathbf{D}=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}^{-1}=\begin{bmatrix}-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\end{bmatrix}.\quad\blacksquare$$

7. Suppose $\mathbf{B}$ is an invertible matrix. Show that $\left(\mathbf{B}^T\right)^{-1} = \left(\mathbf{B}^{-1}\right)^T$. *[10]*

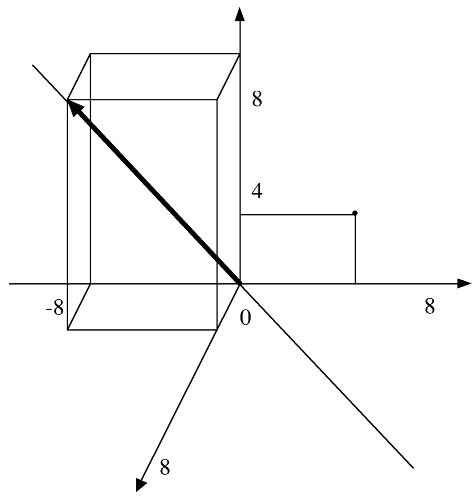
Solution. We will show that $\left(\mathbf{B}^{-1}\right)^T$ acts as the inverse of $\mathbf{B}^T$:

$$\left(\mathbf{B}^{-1}\right)^T \mathbf{B}^T = \left(\mathbf{B}\mathbf{B}^{-1}\right)^T = \mathbf{I}^T = \mathbf{I}$$

Since a matrix may have only one inverse (if any!), it follows that $\left(\mathbf{B}^T\right)^{-1} = \left(\mathbf{B}^{-1}\right)^T$. ■

8. Sketch the point $(0, 5, 3)$ and the line given by the parametric equations $x = 2t$, $y = -6t$, and $z = 10t$, where $t \in \mathbb{R}$, and find the distance between them. *[10]*

Solution. Note that $(0, 5, 3)$ is in the yz-plane (*i.e.* $x = 0$), and that the line passes through the origin – when $t = 0$, we get $(0, 0, 0)$ – and has direction vector $\begin{bmatrix} 2 \\ -6 \\ 10 \end{bmatrix}$. Sketching all this gives:



To find the (shortest!) distance from the given point to the given line, we follow the procedure given in Example 1.25 of the text.

First, let $\mathbf{v}$ be the vector from the point $(0, 0, 0)$ on the line to the given point, $(0, 5, 3)$, namely $\mathbf{v} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$.

Second, we compute the projection of $\mathbf{v}$ onto the direction vector $\mathbf{d} = \begin{bmatrix} 2 \\ -6 \\ 10 \end{bmatrix}$ of the line:

$$\begin{aligned} \mathrm{proj}_{\mathbf{d}}(\mathbf{v}) &= \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}} \right) \mathbf{d} = \frac{0 \cdot 2 + 5 \cdot (-6) + 3 \cdot 3}{2 \cdot 2 + (-6) \cdot (-6) + 3 \cdot 3} \begin{bmatrix} 2 \\ -6 \\ 10 \end{bmatrix} \\ &= -\frac{21}{49} \begin{bmatrix} 2 \\ -6 \\ 10 \end{bmatrix} = -\frac{3}{7} \begin{bmatrix} 2 \\ -6 \\ 10 \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} \\ \frac{18}{7} \\ -\frac{30}{7} \end{bmatrix} \end{aligned}$$

Third, we find the component of $\mathbf{v}$ perpendicular to the line by subtracting the part parallel to the line, $\mathrm{proj}_{\mathbf{d}}(\mathbf{v})$, from $\mathbf{v}$:

$$\mathbf{v} - \mathrm{proj}_{\mathbf{d}}(\mathbf{v}) = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix} - \begin{bmatrix} -\frac{6}{7} \\ \frac{18}{7} \\ -\frac{30}{7} \end{bmatrix} = \begin{bmatrix} \frac{6}{7} \\ \frac{17}{7} \\ \frac{51}{7} \end{bmatrix}$$

Fourth, the desired distance is the length of the component of $\mathbf{v}$ perpendicular to the line:

$$\begin{aligned}\|\mathbf{v} - \operatorname{proj}_{\mathbf{d}}(\mathbf{v})\| = \left\| \begin{bmatrix} \frac{6}{7} \\ \frac{17}{7} \\ \frac{51}{7} \end{bmatrix} \right\| &= \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{17}{7}\right)^2 + \left(\frac{51}{7}\right)^2} \\ &= \sqrt{\frac{36 + 289 + 2601}{49}} = \sqrt{\frac{2926}{49}} = \sqrt{\frac{418}{7}}\end{aligned}$$

Thus the distance from the given point to the given line is $\sqrt{\frac{418}{7}}$. ■

9. Find a linearly independent subset of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ that is as large as possible. *[10]*

Solution. A maximal linearly independent subset of the given set of vectors would be a basis for the span of the given set of vectors. We therefore assemble the vectors into the columns of a matrix and row-reduce it to identify such a basis:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{array}{c} R_2 - R_1 \\ \Longrightarrow \end{array} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{c} R_1 + R_2 \\ \Longrightarrow \\ R_3 + R_2 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{array}{c} \Longrightarrow \\ (-1)R_2 \\ R_3 \leftrightarrow R_4 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 2 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{c} R_2 - R_3 \\ \Longrightarrow \\ R_4 + R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \begin{array}{c} \Longrightarrow \\ \frac{1}{2}R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c} R_1 - R_4 \\ R_2 + R_4 \\ \Longrightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

It follows that the columns of the original matrix corresponding to the columns of the reduced matrix in which leading 1s of a row occur form a basis for the span of the columns, *i.e.* of the original set of vectors. Hence $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a maximal linearly independent subset of the given set of vectors.

Note that this set must be a maximal linearly independent set also because is a set of four linearly independent vectors in $\mathbb{R}^4$ … ■

10. Give examples, if such exist, of invertible 3×3 matrices $\mathbf{P}$ and $\mathbf{Q}$ such that:

- **a.** $\mathbf{P} + \mathbf{Q}$ is invertible. *[2]*
- **b.** $\mathbf{PQ}$ is invertible. *[2]*
- **c.** $\mathbf{P} + \mathbf{Q}$ is not invertible. *[2]*
- **d.** $\mathbf{PQ}$ is not invertible. *[2]*
- **e.** $\mathbf{P}^2 + \mathbf{Q}^T$ is not invertible. *[2]*

Solution to a. Set $\mathbf{P} = \mathbf{Q} = \mathbf{I}_3$. Then both are invertible as the identity matrix is its own inverse, and so is $\mathbf{P} + \mathbf{Q} = 2\mathbf{I}_3$, whose inverse is $\frac{1}{2}\mathbf{I}_3$. ■

Solution to b. Set $\mathbf{P} = \mathbf{Q} = \mathbf{I}_3$. Then both are invertible as the identity matrix is its own inverse, and so is $\mathbf{PQ} = \mathbf{I}_3^2 = \mathbf{I}_3$. ■

Solution to c. Set $\mathbf{P} = \mathbf{I}_3$ and set $\mathbf{Q} = -\mathbf{I}_3$. Then both are invertible as each is its own inverse. However, $\mathbf{P} + \mathbf{Q} = \mathbf{I}_3 + (-\mathbf{I}_3) = \mathbf{0}$ is obviously not invertible. ■

Solution to d. There is no such example in this case, since if $\mathbf{P}$ and $\mathbf{Q}$ are both invertible, so is $\mathbf{PQ}$. Recall that then $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$. ■

Solution to e. Set $\mathbf{P} = \mathbf{I}_3$ and set $\mathbf{Q} = -\mathbf{I}_3$. Then both are invertible as each is its own inverse. However, $\mathbf{P}^2 + \mathbf{Q}^T = \mathbf{I}_3^2 + (-\mathbf{I}_3)^T = \mathbf{I}_3 - \mathbf{I}_3 = \mathbf{0}$ is obviously not invertible. ■

11. Suppose $T : \mathbb{R}^n \to \mathbb{R}^k$ is a linear transformation. Explain why the null space of T, $\text{null}(T) = \{\, \mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0} \,\}$, is a subspace of $\mathbb{R}^n$. *[10]*

Solution. We will show that null(T) satisfies the definition of a subspace.

First, suppose that $\mathbf{x} \in \text{null}(T)$, *i.e.* $T(\mathbf{x}) = \mathbf{0}$, and that c is a scalar. Then $T(c\mathbf{x}) = cT(\mathbf{x}) = c\mathbf{0} = \mathbf{0}$, since linear transformations respect multiplication by scalars, so $c\mathbf{x} \in \text{null}(T)$ too.

Second, suppose that $\mathbf{u} \in \text{null}(T)$ and $\mathbf{v} \in \text{null}(T)$, *i.e.* $T(\mathbf{u}) = \mathbf{0}$ and $T(\mathbf{v}) = \mathbf{0}$. Then $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$, since linear transformations also respect addition of vectors, so $\mathbf{u} + \mathbf{v} \in \text{null}(T)$ too.

Hence null(T) is a subspace of $\mathbb{R}^n$. ■

[Total = 95]

Part Max. Bonus!

Min. Write an original little poem about linear algebra or mathematics in general. *[2]*

Solution. *Left to the reader!* ■

Have a nice break!
I hope to see you next term!