Mathematics 135H – Linear algebra I: matrix algebra
Trent University, Fall 2007

MATH 135H Test Solutions

1. Consider the planes defined by the equations $x + 2y + z = 6$ and $2x + y + z = 6$ in three-dimensional space.

a. Sketch the parts of the two planes and the line in which they intersect that lie in the first octant (*i.e.* where $x \geq 0$, $y \geq 0$, and $z \geq 0$). *[4]*

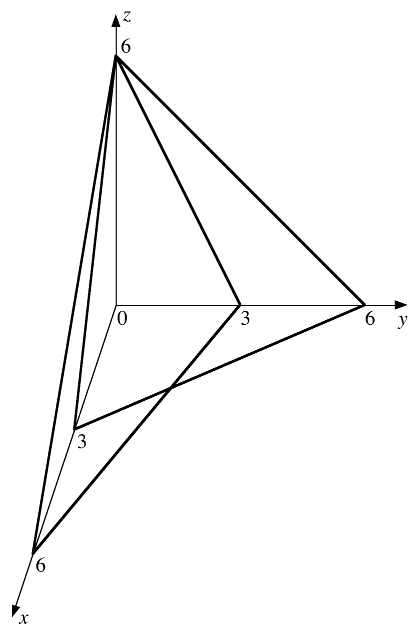
Solution. We first look for any intercepts – intersections with the coordinate axes – the planes may have by setting two of the variables at a time to 0 and solving for the remaining variable in each equation:

i. x-intercepts: If $y = z = 0$, then $x + 2y + z = 6$ gives $x = 6$ and $2x + y + z = 6$ gives $x = 3$.

ii. y-intercepts: If $x = z = 0$, then $x + 2y + z = 6$ gives $y = 3$ and $2x + y + z = 6$ gives $y = 6$.

iii. z-intercepts: If $x = y = 0$, then $x + 2y + z = 6$ gives $z = 6$ and $2x + y + z = 6$ gives $z = 6$ too.

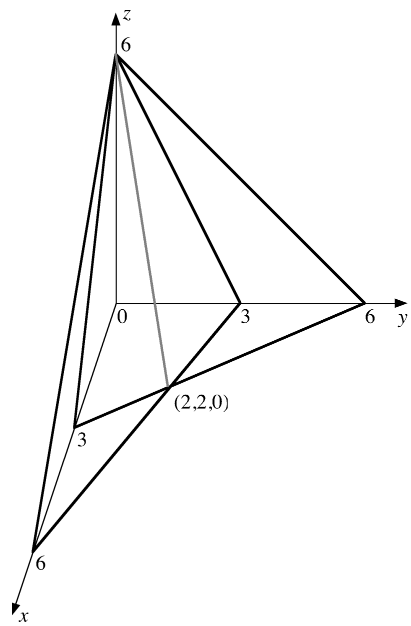
Thus the plane given by $x + 2y + z = 6$ includes the points $(6, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 6)$, while the plane given by $2x + y + z = 6$ includes the points $(3, 0, 0)$, $(0, 6, 0)$, and $(0, 0, 6)$. To draw the part of each plane that lies in the first octant, one simply connects each set of three points.



Note that both planes pass through $(0, 0, 6)$ and that the parts of the planes beyond these triangles are outside the first octant.

To draw the part of the line of intersection in the first octant, observe that the bottom edges of the two triangles (which are lines in the plane $z = 0$) clearly intersect in a point.

(The coordinates of this point are $(2, 2, 0)$ – see the solution to part **b** below – but this doesn't really matter for drawing the picture.) Connect this point, which both planes have in common, in the sketch of the two planes to the common point $(0, 0, 6)$ we already know about.



Note that the parts of the line beyond these points are outside the first octant. ■

b. Give a parametric description of the line in which the two planes intersect. *[3]*

Solution. For a parametric description of the line we need a point on the line and a direction vector parallel to the line. From part **a** it is clear that $(0, 0, 6)$ must be a point on the line.

To find a direction vector it suffices to find another point on the line and take the vector between them. The point where the two planes intersect the xy-plane (*i.e.* $z = 0$) is easy to find: its coordinates are $(x, y, 0)$, where x and y satisfy the equations $x+2y+0 = 6$ and $2x + y + 0 = 6$. Rearranging the first equation, $x = 6 - 2y$; plugging this into the second equation gives $2(6-2y)+y = 6$. Simplifying this a bit gives $-3y = -6$, and solving this in turn for y gives $y = 2$. Plugging $y = 2$ back into, say, $x + 2y + 0 = 6$ and solving for x then yields $x = 2$. Hence the coordinates of the point the given planes intersect the xy-plane are $(2, 2, 0)$. The vector from $(0, 0, 6)$ to $(2, 2, 0)$, $\begin{bmatrix} 2 & 2 & -6 \end{bmatrix}$, will serve as a direction vector for the line.

The vector-parameteric description of the line, written using column vectors, is then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \\ -6 \end{bmatrix}$, where t is the parameter. ■

c. Determine whether the two planes are parallel, perpendicular, or neither. *[3]*

Solution. Since the two planes are clearly different and intersect in a line, they cannot be parallel.

To see if the planes are perpendicular, we check to see if their normal vectors are perpendicular, which we do by checking if the dot product of the two normal vectors is

0. $[1 \quad 2 \quad 1]$ is a vector normal to $x + 2y + z = 6$ and $[2 \quad 1 \quad 1]$ is a vector normal to $2x + y + z = 6$. Since $[1 \quad 2 \quad 1] \cdot [2 \quad 1 \quad 1] = 1 \cdot 2 + 2 \cdot 1 + 1 \cdot 1 = 2 + 2 + 1 = 5 \neq 0$, the two normal vectors, and hence also the two planes, are not perpendicular.

Thus the two planes are neither parallel nor perpendicular. ■

2. Let $\mathbf{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 8 \\ -2 \\ 2 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

a. Determine whether $\mathbf{d} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ is in Span$\{\,\mathbf{a}, \mathbf{b}, \mathbf{c}\,\}$ or not. *[5]*

Solution. $\mathbf{d}$ is in Span$\{\,\mathbf{a}, \mathbf{b}, \mathbf{c}\,\}$ if there are scalars x, y, and z such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$. Working this out coordinate by coordinate, it boils down to checking whether the system of linear equations

$$\begin{array}{rcrcrcr} 3x & + & 8y & - & z & = & 2 \\ x & - & 2y & + & 2z & = & 3 \\ 2x & + & 2y & + & z & = & 3 \end{array}$$

has a solution. We'll do this by using Gaussian elimination on this system in augmented matrix form.

$$\left[\begin{array}{ccc|c} 3 & 8 & -1 & 2 \\ 1 & -2 & 2 & 3 \\ 2 & 2 & 1 & 3 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 3 & 8 & -1 & 2 \\ 2 & 2 & 1 & 3 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 14 & -7 & -7 \\ 0 & 6 & -3 & -3 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ \frac{1}{14}R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 6 & -3 & -3 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ R_3 - 6R_2 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since we now have fewer non-zero rows than there are variables, this system has (infinitely many) solutions. This means that $\mathbf{d}$ can be written as a linear combination of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, so $\mathbf{d}$ is in Span$\{\,\mathbf{a}, \mathbf{b}, \mathbf{c}\,\}$. ■

b. Determine whether $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are linearly dependent or independent. *[5]*

Solution. $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are linearly dependent if there are scalars x, y, and z, not all 0, such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$. Working this out coordinate by coordinate, it boils down to checking whether the system of linear equations

$$\begin{array}{rcrcrcr} 3x & + & 8y & - & z & = & 0 \\ x & - & 2y & + & 2z & = & 0 \\ 2x & + & 2y & + & z & = & 0 \end{array}$$

has a non-trivial solution. Notice that this exactly the same system of equations that came up in part **a**, except that the right-hand side of each equation is 0. We can run through exactly the same steps using Gaussian elimination – note that the right hand sides will remain zero throughout – to get the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since this matrix has fewer non-zero rows than there are variables, this system has infinitely many solutions, and all but one of these solutions must have at least one of the variables be different from 0. Hence **a**, **b**, and **c** are linearly dependent. ■

3. Consider the following system of linear equations.

$$\begin{array}{ccccccc} x & & & + & z & = & 1 \\ x & & & - & z & = & 0 \\ x & + & cy & + & z & = & 0 \end{array}$$

a. Find the solution(s), if any, of the given system in terms of c, the coefficient of y in the third equation. *[6]*

Solution. We set up the system as an augmented matrix and use Gauss-Jordan method:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & c & 1 & 0 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & c & 0 & -1 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ R_3 \leftrightarrow R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & c & 0 & -1 \\ 0 & 0 & -2 & -1 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ \frac{1}{c}R_2 \\ -\frac{1}{2}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -\frac{1}{c} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \begin{array}{c} R_1 - R_3 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{c} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

We can now read off the solution: $x = \frac{1}{2}$, $y = -\frac{1}{c}$, $z = \frac{1}{2}$. Note that c only appears in the value of y. ■

b. For which values of c are there: *i.* No solutions? *ii.* Exactly one solution? *iii.* Many solutions? Explain why in each case. *[4]*

Solution. From the answer to part **a** we know that if there is a solution to the given system of equations, then it is $x = \frac{1}{2}$, $y = -\frac{1}{c}$, $z = \frac{1}{2}$. This solution is unique and makes sense so long as $c \neq 0$, as does the process for finding the solution.

On the other hand, if $c = 0$, the Gauss-Jordan process would terminate at the third augmented matrix, because the second row would then be $0 \;\; 0 \;\; 0 \;\; | \;\; -1$, indicating that there is no solution. One can only continue the process past that stage if one can divide by c. (Remember, dividing by 0 will seriously endanger your health! :-)

Thus there is only one solution to the system if $c \neq 0$, and no solution if $c = 0$. Since these account for all possible values of c, the case of many solutions never arises. ■

4. Let $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 1 & 3 \end{bmatrix}$ and $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In each of **a**–**c**, find an example of a 2×3 matrix **A** satisfying the given matrix equation or explain why there is no such **A**.

a. $\mathbf{AB} = \mathbf{I}_2$. *[4]*

Solution. Suppose $\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Writing out $\mathbf{AB} = \mathbf{I}_2$ gives:

$$\mathbf{AB} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} a+2b+c & -a+3c \\ d+2e+f & -d+3f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2$$

Finding a matrix $\mathbf{A}$ to do this thus boils down to finding solutions to the following systems of linear equations:

$$\begin{array}{rcrcrcr} a & + & 2b & + & c & = & 1 \\ -a & & & + & 3c & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcr} d & + & 2e & + & f & = & 0 \\ -d & & & + & 3f & = & 1 \end{array}$$

One can certainly find such solutions using the general techniques we have learned, but in this case it is probably easier to find such solutions by "hit-or-miss" fiddling. The first system is very easy to satisfy if we make $a = c = 0$ to satisfy the second equation and note that b must then be $\frac{1}{2}$ to make the first one work. To satisfy the second system, it suffices to set $d = -1$ and $f = 0$ to make the second equation work, and then $e = \frac{1}{2}$ to make the first equation work.

Hence $\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -1 & \frac{1}{2} & 0 \end{bmatrix}$ satisfies $\mathbf{AB} = \mathbf{I}_2$. ■

b. $\mathbf{BA} = \mathbf{I}_2$. *[3]*

Solution. There is no such $\mathbf{A}$: since $\mathbf{B}$ is a 3×2 matrix and $\mathbf{A}$ is supposed to be a 2×3 matrix, $\mathbf{BA}$ would have to be a 3×3 matrix, not a 2×2 matrix like $\mathbf{I}_2$. ■

c. $\mathbf{AA}^T = \mathbf{I}_2$. *[3]*

Solution. Suppose $\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Writing out $\mathbf{AA}^T = \mathbf{I}_2$ gives:

$$\mathbf{AA}^T = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} a^2+b^2+c^2 & da+eb+fc \\ ad+be+cf & d^2+e^2+f^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2$$

Finding a matrix $\mathbf{A}$ to do this thus boils down to finding solutions to the following system of four non-linear equations in six variables:

$$\begin{array}{ll} a^2+b^2+c^2 = 1 & ad+be+cf = 0 \\ da+eb+fc = 0 & d^2+e^2+f^2 = 1 \end{array}$$

Note that the second and third of these equations duplicate each other.

This is not a course about solving systems of non-linear equations, so we'll tackle these with "hit-or-miss" fiddling. The first and fourth equations can be satisfied by making one of the three variables involved in each equation 1 and setting the others to 0. This can be done while satisfying the second (and hence also the third) by ensuring that it is non-corresponding variables that are made equal to 1. For example, $a = e = 1$ and $b = c = d = f = 0$ will do the trick.

Hence $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ satisfies $\mathbf{AA}^T = \mathbf{I}_2$. ■

Nikolai Ivanovich Lobachevsky

Who made me what I am today:
the mathematician that others all quote?
Who is the one who made me that way?
The greatest that ever got chalk on his coat!

One man deserves the credit, One man deserves the blame,
And Nikolai Ivanovich Lobachevsky is his name, Hey!
Nikolai Ivanovich Lobach . . .

I am never forget the day I first meet great Lobachevsky;
In one word, he tells me secret to success in mathematics:
Plagiarize!!

Plagiarize!! Let no one else's work evade your eyes!
Remember why the good Lord made your eyes,
So don't shade your eyes,
But plagiarize, plagiarize, plagiarize!!
(Only remember always to call it, please, "research".)

And ever since I meet this man, my life is not the same,
And Nikolai Ivanovich Lobachevsky is his name,
Hey! Nikolai Ivanovich Lobach . . .

I am never forget the day I am given first original paper to write.
It was on algebraic and analytic topology of locally euclidean metrization
Of infinitely differentiable Riemannian manifolds.
Bozhe moi!!
This, I know, from nothing.
But I think of great Lobachevsky and I get idea; hah *hah*!!

I have a friend in Minsk
Who has a friend in Pinsk
With friend in Omsk, with friend in Tomsk
With friend in Akhmolinsk.
His friend in Alexandrovsk
Has friend in Petropavlovsk
Whose friend somehow is solving now
The problem in Dnepropetrovsk.

And when his work is done,
Hee hee!! Begins the fun.
From Dnepropetrovsk to Alexandrovsk
By way of Ysessisk and Novorossisk,
To Omsk, to Tomsk, to Minsk to Pinsk,
To *me*, the news will run,
Yes, to *me* the news will run...

And *then* I write, by morning, night,
And afternoon, and pretty soon
In Dnepropetrovsk, my name is cursed
When he finds out, I published first!!

And who deserves the credit? And who deserves the blame?
Nikolai Ivanovich Lobachevsky is his name, Hey!
Nikolai Ivanovich Lobach . . .

I am never forget the day my first book is published.
Every chapter I steal from somewhere else.
Index I copy from old Vladivostok telephone directory.
This book! This book was sensational!
Pravda – Well, Pravda said [insert nonsense in Russian] – It stinks!!
But Izvestia!! Izvestia said [insert more nonsense in Russian] – It stinks!!

Metro Goldwyn Moskva buys film rights for 40 million rubles,
And changes name to *The Eternal Triangle*
With Doris Day playing part hypotenuse!

[Chorus, this time actually finishing.]

Tom Lehrer