Mathematics 135H – Linear algebra I: matrix algebra

Trent University, Fall 2007

Solutions to Assignment #5

1. Find the matrix R_θ^z of a rotation through an angle of θ about the z-axis. *[1]*

Note: This rotation leaves the z-coordinate unchanged. As with rotations about the origin in $\mathbb{R}^2$, θ is measured counterclockwise, starting with the positive x-axis, when the xy-plane is viewed from above (*i.e.* from the positive z-axis).

Solution. Since the matrix leaves the z-coordinate unchanged and the z-coordinate should not affect what the matrix does to the x- and y-coordinates, the third row and the third column must look like

$$\begin{bmatrix} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In terms of the x- and y-coordinates, this matrix behaves just like a rotation through an angle of θ about the origin in $\mathbb{R}^2$. Filling the missing part of the matrix in accordinagly gives

$$R_\theta^z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \blacksquare$$

2. Find the matrix R_ϕ^y of a rotation through an angle of ϕ about the y-axis. *[1]*

Note: This rotation leaves the y-coordinate unchanged. The angle ϕ should be measured counterclockwise, starting with the positive x-axis, when the xz-plane is viewed from the positive y-axis.

Solution. This is almost like Problem **1** above, the obvious exceptions being calling the angle ϕ instead of θ and the interchanging the roles of the variables y and z. Thus the first cut at R_ϕ^y would probably be:

$$\begin{bmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

The problem is that this matrix is for a rotation in the wrong direction: as viewed from the positive y-axis, it rotates things clockwise about the y-axis, rather than counterclockwise. Consider, for example, a rotation of $\phi = 45°$ by this matrix. $\cos(45°) = \sin(45°) = \frac{1}{\sqrt{2}}$, so in this case the matrix would be:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Note, however, that

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

which is *not* counterclockwise ...

This problem can be fixed by plugging in the negative of the desired angle into the matrix above. Thus, since $\cos(-\phi) = \cos(\phi)$ and $\sin(-\phi) = -\sin(\phi)$,

$$R^y_\phi = \begin{bmatrix} \cos(-\phi) & 0 & -\sin(-\phi) \\ 0 & 1 & 0 \\ \sin(-\phi) & 0 & \cos(-\phi) \end{bmatrix} = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

is the matrix we want. ■

Note: For those of you who are into linear transformations, the matrix of the linear transformation that swaps the roles of y and z and reverses orientation along the y-axis (which is basically what we had to do above) is $\mathbf{S}_{yz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. One way to get R^y_ϕ is to do $\mathbf{S}_{yz}$ first, which swaps the y- and z-axes and reverses orientation along the y-axis, do a rotation through an angle ϕ about the (new) z-axis by doing R^z_ϕ, and then restore the original y- and z-axes by doing $\mathbf{S}_{yz}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. You can check for yourself that this works by multiplying things out below:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

3. Find the matrix R^x_α of a rotation through an angle of α about the x-axis. *[1]*

Note: This rotation leaves the x-coordinate unchanged. The angle α should be measured counterclockwise, starting with the positive y-axis, when the yz-plane is viewed from the positive x-axis.

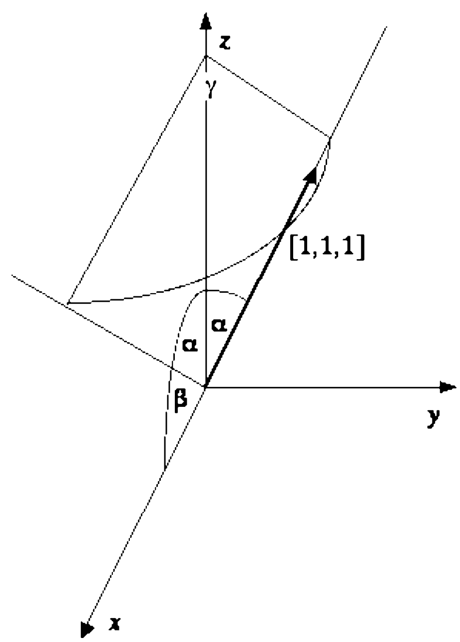
Solution. This is just like Problem **2** above except that we start with R^y_ϕ instead of R^z_θ, we're calling the angle α instead of ϕ, the variables x and y exchange roles, and orientation must be reversed again. To cut to the chase,

$$R^x_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}. \qquad ■$$

4. Find a combination of the rotations you obtained in **1–3** that moves the x-axis onto the line through the origin with direction vector $[1 \quad 1 \quad 1]$. *[2]*

Solution. We will move the x-axis onto the line through the origin with direction vector $[1 \quad 1 \quad 1]$ in stages. First, we will do a rotation about the y-axis that moves the x-axis to another line in the xz-plane, and then follow this with a rotation about the z-axis that will move this line to the line with direction vector $[1 \quad 1 \quad 1]$.

The key to making this strategy work is figuring out what the intermediate line needs to be, and the key to that is the observation that a rotation about the z-axis must preserve the angle that a line though the origin makes with the z-axis. Thus the intermedite line must make the same angle, call it α, with the z-axis that the line with direction vector $[1 \quad 1 \quad 1]$ does.



We can, in principle, compute α pretty easily by computing the angle between direction vectors, $[1 \quad 1 \quad 1]$ for our target line and, say, $[0 \quad 0 \quad 1]$ for the z-axis:

$$\cos(\alpha) = \frac{[1 \quad 1 \quad 1] \cdot [0 \quad 0 \quad 1]}{\| [1 \quad 1 \quad 1] \| \, \| [0 \quad 0 \quad 1] \|} = \frac{1 \cdot 0 + 1 \cdot 0 + 1 \cdot 1}{\sqrt{1^2 + 1^2 + 1^2}\,\sqrt{0^2 + 0^2 + 1^2}} = \frac{1}{\sqrt{3}}$$

From this, with the use of a calculator, we can compute that that $\alpha \approx 54.74°$.

The first of our rotations, about the y-axis, must take the positive x-axis into the line through the origin in the xz-plane between the positive x- and z-axes that makes an angle α with the positive z-axis. To accomplish this will require a rotation about the y-axis through an angle of $\beta = 90° - \alpha$ from the positive x-axis in the direction of the positive z-axis. Note that this is a rotation clockwise as seen from the positive y-axis, so we must plug in $\phi = -\beta = \alpha - 90°$ into the matrix R^y_ϕ obtained in **2**.

To actually work the matrix R^y_ϕ out, we need to compute $\cos(\phi)$ and $\sin(\phi)$. To do this we will use the difference formulas for cos and sin, namely $\cos(a - b) = \cos(a)\cos(b) +$

$\sin(a)\sin(b)$ and $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$, and the fact that $\sin^2(c) = \sqrt{1-\cos^2(c)}$.

$$\begin{aligned}\cos(\phi) &= \cos(\alpha - 90^\circ) = \cos(\alpha)\cos(90^\circ) + \sin(\alpha)\sin(90^\circ) \\ &= \frac{1}{\sqrt{3}} \cdot 0 + \sqrt{1 - \left(\frac{1}{\sqrt{3}}\right)^2} \cdot 1 = \frac{\sqrt{2}}{\sqrt{3}}\end{aligned}$$

$$\begin{aligned}\sin(\phi) &= \sin(\alpha - 90^\circ) = \sin(\alpha)\cos(90^\circ) - \cos(\alpha)\sin(90^\circ) \\ &= \frac{\sqrt{2}}{\sqrt{3}} \cdot 0 - \frac{1}{\sqrt{3}} \cdot 1 = -\frac{1}{\sqrt{3}}\end{aligned}$$

Thus the matrix we want for the rotation about the y-axis is:

$$R^y_\phi = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

The matrix R^y_ϕ obtained above moves the line with direction vector $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ (*i.e.* the x-axis) to the line with direction vector $\begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$ via a rotation about the y-axis. It remains to move the latter line to the line with direction vector $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ via a suitable rotation about the z-axis. Again, the key is to determine the necessary angle, θ, of the rotation.

The desired rotation keeps the z-axis fixed, so it must move the plane determined by the lines with direction vectors $\begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ (*i.e.* the z-axis) to the plane determined by the lines with direction vectors $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ (*i.e.* the z-axis). The acute angle between the planes – which is the angle *theta* we wish to determine – is the same as the acute angle between their normal vectors. We can find the normal vector to each plane by taking the cross-product of the two direction vectors in that plane:

$$\begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 1 \end{vmatrix} = -\frac{\sqrt{2}}{\sqrt{3}}\mathbf{j} = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{\sqrt{3}} & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}.$$

The acute angle θ between the normal vectors can now be determined:

$$\cos(\theta) = \frac{\begin{bmatrix} 0 & -\frac{\sqrt{2}}{\sqrt{3}} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}}{\| \begin{bmatrix} 0 & -\frac{\sqrt{2}}{\sqrt{3}} & 0 \end{bmatrix} \| \, \| \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \|} = \frac{\frac{\sqrt{2}}{\sqrt{3}}}{\frac{\sqrt{2}}{\sqrt{3}} \cdot \sqrt{2}} = \frac{1}{\sqrt{2}}$$

It follows that $\theta = 45°$.

Thus the matrix we want for the rotation about the z-axis is:

$$R^z_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that a combination of rotations that moves the x-axis onto the line through the origin with direction vector $[1 \quad 1 \quad 1]$ is:

$$R^z_\theta R^y_\phi = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

Note that the rotation which goes first goes on the right of the matrix product.

We can check that this does the job by checking where a direction vector of the positive x-axis, say $[1 \quad 0 \quad 0]$, goes:

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Since $\left[\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}}\right]$ is parallel to $[1 \quad 1 \quad 1]$, $R^z_\theta R^y_\phi$ does indeed move the x-axis onto the line through the origin with direction vector $[1 \quad 1 \quad 1]$, as desired. ■

5. Find a combination of the rotations you obtained in **1**–**3** that moves the line through the origin with direction vector $[1 \quad 1 \quad 1]$ onto the x-axis. *[1]*

Solution. Here all we need to do is what we did in **4** in reverse: instead of doing $R^z_\theta R^y_\phi$ as in **4**, we do $R^y_{-\phi} R^z_{-\theta}$. (Note that we need to reverse the order in which we do the rotations as well as reverse the rotations themselves.) Since we already know R^z_θ and R^y_ϕ from **4**, we can easily work out $R^y_{-\phi}$ and $R^z_{-\theta}$ using the relations $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$.

First, since

$$R^z_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we have

$$R^z_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Second, since

$$R^y_\phi = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix},$$

we have

$$R^y_{-\phi} = \begin{bmatrix} \cos(-\phi) & 0 & \sin(-\phi) \\ 0 & 1 & 0 \\ -\sin(-\phi) & 0 & \cos(-\phi) \end{bmatrix} = \begin{bmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}.$$

It follows that a combination of rotations that moves the line through the origin with direction vector $[1 \quad 1 \quad 1]$ onto the x-axis is:

$$R^y_{-\phi} R^z_{-\theta} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

It is not hard to check that

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 0 \\ 0 \end{bmatrix},$$

which is a vector parallel to $[1 \quad 0 \quad 0]$, as desired. ■

6. Find the matrix R of a rotation through an angle of ω about the line through the origin with direction vector $[1 \quad 1 \quad 1]$. *[4]*

Note: The angle ω should be measured counterclockwise when the plane $x + y + z = 1$ is viewed from the first octant.

Hint: Put together **3**–**5**.

Solution. The strategy here is similar to that described in the note after the solution to **2**: move the line with direction vector $[1 \quad 1 \quad 1]$ to the x-axis (as in **5**), execute a rotation of ω about the x-axis (as in **3**), and then move the x-axis back to the line with direction vector $[1 \quad 1 \quad 1]$ to the x-axis (as in **4**). Thus

$$\begin{aligned} R &= \left(R^z_\theta R^y_\phi\right) R^x_\omega \left(R^y_{-\phi} R^z_{-\theta}\right) \\ &= \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega) & -\sin(\omega) \\ 0 & \sin(\omega) & \cos(\omega) \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}, \end{aligned}$$

and we leave it to the reader to multiply the matrices out to get R ... ■