Mathematics 135H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2007

Solutions to Assignment #4

Recall the definition given in Assignment #4:

⋆ A $k \times k$ matrix $\mathbf{B}$ *absorbs* the $k \times k$ matrix $\mathbf{A}$ if $\mathbf{B}\mathbf{A}^m = \mathbf{A}^m\mathbf{B} = \mathbf{B}$ for every $m > 0$.

The key to much of what follows is that this definition boils down to something a little simpler:

- A $k \times k$ matrix $\mathbf{B}$ absorbs a $k \times k$ matrix $\mathbf{A}$ exactly if $\mathbf{BA} = \mathbf{AB} = \mathbf{B}$.

It's not hard to see why this is so. First, if the original definition is true, $\mathbf{B}\mathbf{A}^m = \mathbf{A}^m\mathbf{B} = \mathbf{B}$ must be true for $m = 1$ in particular. Second, if $\mathbf{BA} = \mathbf{AB} = \mathbf{B}$ is true, then, for example, $\mathbf{BA}^2 = \mathbf{BAA} = \mathbf{BAB}$. It's not hard to see that it must follow from such reasoning that $\mathbf{B}\mathbf{A}^m = \mathbf{A}^m\mathbf{B} = \mathbf{B}$ for every $m > 0$.

Using the simpler version of the definition reduces the amount of work required in some of the solutions below.

1. Verify that $\mathbf{0}_k$ absorbs $\mathbf{A}$, for any $k \times k$ matrix $\mathbf{A}$. *[2]*

Solution. We noted in class that if $\mathbf{A}$ is a $k \times k$ matrix, then $\mathbf{A}\mathbf{0}_k = \mathbf{0}_k\mathbf{A} = \mathbf{0}_k$ (among other algebraic properties of matrix multiplication). By the simpler definition of absorbtion given above, this means that $\mathbf{0}_k$ absorbs $\mathbf{A}$ for any $k \times k$ matrix $\mathbf{A}$. ■

Note: If you need to convince yourself that $\mathbf{A}\mathbf{0}_k = \mathbf{0}_k\mathbf{A} = \mathbf{0}_k$, note first that $0\mathbf{B} = \mathbf{0}_k$ for any $k \times k$ matrix $\mathbf{B}$. Then, for example, $\mathbf{A}\mathbf{0}_k = \mathbf{A}(0\mathbf{0}_k) = 0(\mathbf{A}\mathbf{0}_k) = \mathbf{0}_k$.

2. Find an example of a $k \times k$ matrix $\mathbf{A} \neq \mathbf{0}_k$ such that $\mathbf{A}$ absorbs itself. *[2]*

Solution. $\mathbf{I}_k$ does the job by the simpler definition of absorbtion, because $\mathbf{I}_k\mathbf{I}_k = \mathbf{I}_k$. ■

3. Find an example of a $k \times k$ matrix $\mathbf{A} \neq \mathbf{0}_k$ such that $\mathbf{0}_k$ is the *only* $k \times k$ matrix that absorbs $\mathbf{A}$. *[3]*

Solution. We'll do this for $k = 2$, though the gimmick we'll use is easily adaptable to any $k \geq 2$. Note that by Problem **1**, $\mathbf{0}_2$ absorbs any 2×2 matrix.

Let $\mathbf{A} = \frac{1}{2}\mathbf{I}_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, and suppose that $\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix that absorbs $\mathbf{A}$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{B} = \mathbf{AB} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{a}{2} & \frac{b}{2} \\ \frac{c}{2} & \frac{d}{2} \end{bmatrix},$$

so we must have $a = \frac{a}{2}$, $b = \frac{b}{2}$, $c = \frac{c}{2}$, and $d = \frac{d}{2}$. This can only occur if $a = b = c = d = 0$, so it must be the case that if $\mathbf{B}$ absorbs $\mathbf{A}$, then $\mathbf{B} = \mathbf{0}_2$. ■

4. Suppose the $\mathbf{A}$ is a $k \times k$ matrix which is absorbed by a matrix $\mathbf{B}$ which has an inverse. Show that it must be the case that $\mathbf{A} = \mathbf{I}_k$. *[3]*

Solution. Suppose the $\mathbf{A}$ is a $k \times k$ matrix which is absorbed by a matrix $\mathbf{B}$ that has an inverse. By the simpler definition of absorbtion, this means that $\mathbf{AB} = \mathbf{B}$. Multiplying this equation on both sides by $\mathbf{B}^{-1}$ (from the right) gives:

$$\mathbf{A} = \mathbf{A}\mathbf{I}_k = \mathbf{A}\left(\mathbf{B}\mathbf{B}^{-1}\right) = (\mathbf{AB})\,\mathbf{B}^{-1} = \mathbf{B}\mathbf{B}^{-1} = \mathbf{I}_k$$

Hence $\mathbf{A} = \mathbf{I}_k$, as desired. ■