Mathematics 135H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2007

Solutions to Quizzes

Quiz #1. Friday, 21 September, 2007. *[5 minutes]*

1. Find the acute angle between the vectors $\mathbf{a} = [2, 1, 0]$ and $\mathbf{b} = \left[2, 1, \sqrt{5}\right]$. *[5]*

Solution. Suppose θ is the acute angle between $\mathbf{a}$ and $\mathbf{b}$. Then

$$\begin{aligned}\cos(\theta) &= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{[2,1,0]\cdot\left[2,1,\sqrt{5}\right]}{\|[2,1,0]\|\,\left\|\left[2,1,\sqrt{5}\right]\right\|} \\ &= \frac{2\cdot 2 + 1\cdot 1 + 0\cdot\sqrt{5}}{\sqrt{2^2+1^2+0^2}\sqrt{2^2+1^2+\left(\sqrt{5}\right)^2}} \\ &= \frac{5}{\sqrt{5}\sqrt{10}} = \frac{5}{\sqrt{5}\sqrt{5}\sqrt{2}} = \frac{1}{\sqrt{2}}\,,\end{aligned}$$

so $\theta = 45°$ or $\theta = \frac{\pi}{4}$ radians. ■

Quiz #2. Friday, 28 September, 2007. *[10 minutes]*

1. Find a linear equation $ax + by + cz = d$ of the plane containing both of the lines given by the parametric equations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 7 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 7 \end{bmatrix} + s\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

(Note that both of these lines pass through the point $(0, 6, 7)$.) *[5]*

Solution. To obtain the normal vector $[a, b, c]$ of the plane we need a vector which is perpendicular to the direction vectors of both lines. The cross product of the direction vectors will do:

$$\begin{aligned}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ -1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix}\mathbf{i} + \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}\mathbf{k} \\ &= (0-4)\mathbf{i} - (1-(-2))\,\mathbf{j} + (2-0)\mathbf{k} = -4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} = \begin{bmatrix} -4 \\ -3 \\ 2 \end{bmatrix}\end{aligned}$$

An equation for the plane is therefore $-4x - 3y + 2z = d$. To determine d note that the plane containing both lines must also pass through the point $(0, 6, 7)$, so

$$d = -4\cdot 0 - 3\cdot 6 + 2\cdot 7 = 0 - 18 + 14 = -4\,.$$

Hence a linear equation of the plane containing both of the given lines is

$$-4x - 3y + 2z = -4 . \quad \blacksquare$$

Quiz #3. Friday, 5 Octoberber, 2007. *[10 minutes]*

1. Solve the following system of linear equations. *[5]*

$$\begin{aligned} x + y + z &= 12 \\ x - y + 2z &= 18 \\ 2x + 3y - z &= 24 \end{aligned}$$

Solution. We'll set up the given system of equations in augmented matrix form and solve it using Gauss-Jordan elimination. To save some space, we'll do two row operations at a time when we can safely do so.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 1 & -1 & 2 & 18 \\ 2 & 3 & -1 & 24 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & -2 & 1 & 6 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \Longrightarrow \\ R_2 \leftrightarrow R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & -3 & 0 \\ 0 & -2 & 1 & 6 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \Longrightarrow \\ R_3 + 2R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 12 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -5 & 6 \end{array}\right]$$

$$\begin{array}{c} \\ \Longrightarrow \\ -\frac{1}{5}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 12 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -\frac{6}{5} \end{array}\right] \begin{array}{c} R_1 - 4R_3 \\ R_2 + 3R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{84}{5} \\ 0 & 1 & 0 & -\frac{18}{5} \\ 0 & 0 & 1 & -\frac{6}{5} \end{array}\right]$$

We can now read off the solution from the final augmented matrix: $x = \frac{84}{5}$, $y = -\frac{18}{5}$, and $z = -\frac{6}{5}$. $\blacksquare$

Quiz #4. Friday, 12 Octoberber, 2007. *[10 minutes]*

1. Determine whether $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Show your reasoning. *[5]*

Solution I. By hit and miss fiddling, or however, observe that:

$$\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

It follows that $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ is in the span of $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and hence is also in the span of all three of $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. $\blacksquare$

Solution II. More systematically, note that, by definition, $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ is in the span of $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ if there are scalars a, b, and c such that:

$$a \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

This boils down to checking if there is a solution to the following system of linear equations:

$$\begin{array}{ccccccc} & & b & + & c & = & 2 \\ a & & & + & c & = & 4 \\ a & + & b & & & = & 6 \end{array}$$

We'll set up the given system of equations in augmented matrix form and solve it using Gaussian elimination and back-substitution. Here goes:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 6 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 6 \end{array}\right] \underset{R_3 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\underset{R_3 - R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 0 \end{array}\right] \underset{-\frac{1}{2}R_3}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The last augmented matrix corresponds to the system of linear equations

$$\begin{array}{ccccccc} a & & & + & c & = & 4 \\ & & b & + & c & = & 2 \\ & & & & c & = & 0 \end{array}$$

which we solve by back substitution. Plugging $c = 0$ into $b + c = 2$ gives $b = 2$, and then plugging $c = 0$ and $b = 2$ into $a + c = 4$ gives $a = 4$.

Since the system of linear equations does have a solution, $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ is indeed in the span of $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. ■

Quiz #5. Friday, 19 Octoberber, 2007. *[10 minutes]*

1. Compute $(\mathbf{AB})^T$ if $\mathbf{A} = \begin{bmatrix} 6 & -3 \\ -1 & 0 \\ 2 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 & -4 \\ 0 & -1 & 1 \end{bmatrix}$. *[5]*

Solution. We first compute $\mathbf{AB}$. Note that since $\mathbf{A}$ is a 3×2 matrix and $\mathbf{B}$ is a 2×3 matrix, $\mathbf{AB}$ must be a 3×3 matrix.

$$\begin{aligned}
\mathbf{AB} &= \begin{bmatrix} 6 & -3 \\ -1 & 0 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -4 \\ 0 & -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 6 \cdot 1 + (-3) \cdot 0 & 6 \cdot 2 + (-3) \cdot (-1) & 6 \cdot (-4) + (-3) \cdot 1 \\ (-1) \cdot 1 + 0 \cdot 0 & (-1) \cdot 2 + 0 \cdot (-1) & (-1) \cdot (-4) + 0 \cdot 1 \\ 2 \cdot 1 + 5 \cdot 0 & 2 \cdot 2 + 5 \cdot (-1) & 2 \cdot (-4) + 5 \cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 15 & -27 \\ -1 & -2 & 4 \\ 2 & -1 & -3 \end{bmatrix}
\end{aligned}$$

We next compute $(\mathbf{AB})^T$. Note that since $\mathbf{AB}$ is a 3×3 matrix, $(\mathbf{AB})^T$ must also be a 3×3 matrix.

$$(\mathbf{AB})^T = \begin{bmatrix} 6 & 15 & -27 \\ -1 & -2 & 4 \\ 2 & -1 & -3 \end{bmatrix}^T = \begin{bmatrix} 6 & -1 & 2 \\ 15 & -2 & -1 \\ -27 & 4 & -3 \end{bmatrix} \qquad \blacksquare$$

Quiz #6. Friday, 9 November, 2007. *[10 minutes]*

1. Find the inverse matrix, if it exists, of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. *[5]*

Solution. We set up the appropriate super-augmented matrix and use the Gauss-Jordan method:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \underset{R_3 - R_1}{\Longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R_3 \leftrightarrow R_1}{\Longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \underset{R_4 - R_3}{\Longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

Thus the inverse exists and $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$. $\blacksquare$

Quiz #7. Friday, 16 November, 2007. *[10 minutes]*

1. Suppose $\mathbf{A}$ and $\mathbf{B}$ are invertible $k \times k$ matrices. Solve the matrix equation

$$\left(\mathbf{X}^{-1}\mathbf{A}\right)^{-1} = \mathbf{A}\left(\mathbf{B}^2\mathbf{A}\right)^{-1}$$

for the (invertible) $k \times k$ matrix $\mathbf{X}$. Simplify your answer as much as possible. *[5]*

Solution. We work to isolate $\mathbf{X}$. The first thing to do is to solve for $\mathbf{X}^{-1}\mathbf{A}$:

$$\begin{aligned}\mathbf{X}^{-1}\mathbf{A} &= \left(\left(\mathbf{X}^{-1}\mathbf{A}\right)^{-1}\right)^{-1} = \left(\mathbf{A}\left(\mathbf{B}^2\mathbf{A}\right)^{-1}\right)^{-1} \\ &= \left(\left(\mathbf{B}^2\mathbf{A}\right)^{-1}\right)^{-1}\mathbf{A}^{-1} = \mathbf{B}^2\mathbf{A}\mathbf{A}^{-1} = \mathbf{B}^2\end{aligned}$$

It follows that

$$\mathbf{X}^{-1} = \mathbf{X}^{-1}\mathbf{A}\mathbf{A}^{-1} = \mathbf{B}^2\mathbf{A}^{-1},$$

so

$$\mathbf{X} = \left(\mathbf{X}^{-1}\right)^{-1} = \left(\mathbf{B}^2\mathbf{A}^{-1}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{-1}\left(\mathbf{B}^2\right)^{-1} = \mathbf{A}\mathbf{B}^{-2}.$$

The relation $\mathbf{X} = \mathbf{A}\mathbf{B}^{-2}$ is as simple as it's going to get without further information about $\mathbf{A}$ and $\mathbf{B}$. ■

Quiz #8. Friday, 23 November, 2007. *[10 minutes]*

1. Let $\mathbf{A} = \begin{bmatrix} 5 & 1 & -1 \\ 7 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix}$. Find bases for row($\mathbf{A}$), col($\mathbf{A}$), and null($\mathbf{A}$). *[5]*

Solution. Following the all-in-one approach done in class, we'll do Gauss-Jordan elimination on the augmented matrix representing the homogeneous system $\mathbf{Ax} = \mathbf{0}$.

$$\left[\begin{array}{ccc|c} 5 & 1 & -1 & 0 \\ 7 & 2 & -1 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right] \overset{\frac{1}{5}R_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & \frac{1}{5} & -\frac{1}{5} & 0 \\ 7 & 2 & -1 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right] \underset{R_2 - 7R_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & \frac{1}{5} & -\frac{1}{5} & 0 \\ 0 & \frac{3}{5} & \frac{2}{5} & 0 \\ 0 & 3 & 2 & 0 \end{array}\right]$$

$$\underset{\frac{5}{3}R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & \frac{1}{5} & -\frac{1}{5} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 3 & 2 & 0 \end{array}\right] \begin{array}{c} R_1 - \frac{1}{5}R2 \\ \Longrightarrow \\ R_3 - 3R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The rows of the (coefficient part of the) reduced matrix give a basis for the row space of the original matrix, so $\left\{ \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{2}{3} \end{bmatrix} \right\}$ is a basis for row($\mathbf{A}$).

The columns of the reduced matrix which contain leading 1s of rows indicate columns of the original matrix which make up a basis for the column space of the original matrix, so $\left\{ \begin{bmatrix} 5 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis for col($\mathbf{A}$).

Finally, the reduced augmented matrix corresponds to the system of equations:

$$\begin{array}{ccccccc} x & & & - & \frac{1}{3}z & = & 0 \\ & & y & + & \frac{2}{3}z & = & 0 \end{array}$$

Using t as a parameter and setting $z = t$, it follows that $x = \frac{1}{3}t$ and $y = -\frac{2}{3}t$. Thus the solutions to the homogeneous system $\mathbf{Ax} = \mathbf{0}$ can be written in vector-parameteric form as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Hence $\left\{ \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix} \right\}$ is a basis for null($\mathbf{A}$). ■

Quiz #9. Friday, 30 November, 2007. *[10 minutes]*

1. Find the eigenvalues of $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$. *[5]*

Solution. We need to find the values of λ for which there is a vector $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ such that $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$. This boils down to finding the values of λ such that the system of equations

$$\begin{array}{ccccc} x & & & = & \lambda x \\ x & + & 2y & = & \lambda y \end{array} \quad , \text{ i.e. } \quad \begin{array}{ccccc} (1-\lambda)x & & & = & 0 \\ x & + & (2-\lambda)y & = & 0 \end{array} \quad ,$$

has a non-zero solution. We do this by reducing the augmented matrix of the homogeneous system as far as we can:

$$\begin{aligned} & \left[\begin{array}{cc|c} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \end{array}\right] \\ \begin{array}{c} R_1 \leftrightarrow R_2 \\ \Longrightarrow \end{array} & \left[\begin{array}{cc|c} 1 & 2-\lambda & 0 \\ 1-\lambda & 0 & 0 \end{array}\right] \\ \begin{array}{c} R_2 - (1-\lambda)R_2 \\ \Longrightarrow \end{array} & \left[\begin{array}{cc|c} 1 & 2-\lambda & 0 \\ 0 & -(1-\lambda)(2-\lambda) & 0 \end{array}\right] \end{aligned}$$

At this point it is apparent that if $(1-\lambda)(2-\lambda) \neq 0$, the system has only the solution $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which means that no such λ is an eigenvalue of the given matrix.

On the other hand, if $(1-\lambda)(2-\lambda) = 0$, the system has infinitely many solutions – all but one of which must satisfy $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ – so any such λ is an eigenvalue of the given matrix. Since $(1-\lambda)(2-\lambda) = 0$ only for $\lambda = 1$ and $\lambda = 2$, these are the eigenvalues of the given matrix. ■

Quiz #10. Thursday, 6 December, 2007. *[10 minutes]*

1. Find the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. *[5]*

Solution. We will row-reduce $\mathbf{A}$ to upper-triangular form to compute $|\mathbf{A}|$.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} \Longrightarrow \\ \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\begin{matrix} \Longrightarrow \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \begin{matrix} \Longrightarrow \\ -\frac{1}{2}R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$$\begin{matrix} \Longrightarrow \\ \\ \\ R_4 + R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix}$$

The only row operation we used that would affect the determinant was the multiplication of row 3 by $-\frac{1}{2}$. Hence

$$\left(-\frac{1}{2}\right)|\mathbf{A}| = \left| \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix} \right| = 1 \cdot 1 \cdot 1 \cdot \left(-\frac{3}{2}\right) = -\frac{3}{2},$$

and solving for $|\mathbf{A}|$ gives $|\mathbf{A}| = \left(-\frac{3}{2}\right) \div \left(-\frac{1}{2}\right) = 3$. ■