Mathematics 135H – Linear algebra I: matrix algebra

Trent University, Fall 2007

Final Examination

Monday, 10 December, 2007

Time: 3 hours *Brought to you by* Stefan Bilaniuk.

Instructions: Show all your work. *If in doubt about something,* **ask!**

Aids: Calculator; annotated *Formula for Success* or $8.5'' \times 11''$ aid sheet; one brain.

Part I. Do all of **1**–**5**.

1. Consider the planes in $\mathbb{R}^3$ given by the equations $2x+3y+3z=12$ and $6x+4y+3z=24$, respectively.

a. Sketch the parts of these planes, and their line of intersection, that lie in the first octant. *[5]*

b. Find a parametric description of the line of intersection of the two planes. *[5]*

2. Consider the following system of linear equations.

$$\begin{array}{ccccccccc} w & - & x & - & y & + & z & = & 0 \\ & & x & + & y & & & = & -1 \\ w & + & x & + & 2y & + & z & = & 1 \\ w & + & x & & & + & z & = & 2 \end{array}$$

a. Use Gaussian elimination to find all the solutions, if any, of this system. *[10]*

b. Use your work for **a** to compute the determinant of the coefficient matrix. *[5]*

3. Find the inverse, if it exists, of $\begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$. *[10]*

4. Let $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & -1 \end{bmatrix}$.

a. Use Gauss-Jordan elimination to put $\mathbf{A}$ in reduced echelon form. *[5]*

b. Find bases for row($\mathbf{A}$) and col($\mathbf{A}$), the row and column spaces of $\mathbf{A}$. *[4]*

c. What are the rank and nullity of $\mathbf{A}$? *[1]*

d. Find a basis for null($\mathbf{A}$), the null space of $\mathbf{A}$. *[5]*

5. Find all the eigenvalues of $\mathbf{B} = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}$, and find an eigenvector for each of the eigenvalues. *[15]*

Part II. Do any *three* of **6–11**.

6. Compute the determinant of

$$\mathbf{C} = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ -3 & 4 & 2 & -1 \end{bmatrix}$$

and use it to determine whether $\mathbf{C}$ is invertible or not. *[10]*

7. Suppose $\mathbf{A}$ is a square matrix. What is $\det\left(\mathbf{A}^T\right) = \left|\mathbf{A}^T\right|$ in terms of $\det(\mathbf{A}) = |\mathbf{A}|$? Explain why! *[10]*

8. Find all 2×2 matrices $\mathbf{X}$ satisfying the matrix equation $\mathbf{X}^2 + \mathbf{X} - 2\mathbf{I}_2 = \mathbf{O}_2$. *[10]*

9. Suppose T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}, \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}.$$

What is the matrix $\mathbf{A}_T$ associated to this linear transformation? (This matrix is called the *standard matrix of* T and denoted by $[T]$ in the text.) *[10]*

10. Find a basis for the subspace $S = \text{Span}\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$. *[10]*

11. Find the distance from the point $(2, 0, 1)$ in $\mathbb{R}^3$ to the plane given by the equation $x - y - z = -1$. *[10]*

[Total = 95]

Part Null. Bonus!

0^{0^0}. Write an original little poem about linear algebra or mathematics in general. *[2]*

HAVE A NICE BREAK!
I HOPE TO SEE YOU NEXT TERM!