Mathematics 135H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2007

SOLUTIONS TO THE FINAL EXAMINATION

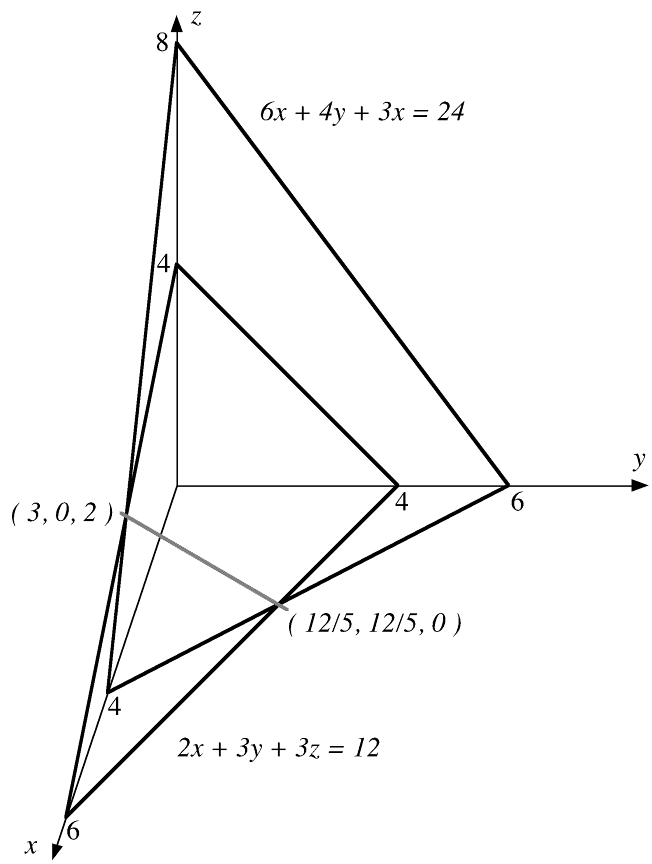
1. Consider the planes in $\mathbb{R}^3$ given by the equations $2x+3y+3z = 12$ and $6x+4y+3z = 24$, respectively.

a. Sketch the parts of these planes, and their line of intersection, that lie in the first octant. *[5]*

Solution. First, we find the intercepts of each of the planes:

	$2x + 3y + 3z = 12$	$6x + 4y + 3z = 24$
$y = z = 0$	$x = 12/2 = 6$	$x = 24/6 = 4$
$x = z = 0$	$y = 12/3 = 4$	$y = 24/4 = 6$
$x = y = 0$	$z = 12/3 = 4$	$z = 24/3 = 8$

To sketch the parts of the planes that lie in the first octant, plot the intercepts and connect up those belonging to each plane. To sketch the part of the line of intersection that lies in the first octant, note the points in the xy- and xz-planes where the edges of the two triangles you just drew meet and connect those up too.



Note that we don't really have to work out the coordinates of the two points we connected up to draw the line of intersection in order to add it to the sketch. ■

b. Find a parametric description of the line of intersection of the two planes. *[5]*

Solution. A parametric description of the line requires a base point on the line and a direction vector.

First, we find the coordinates of one of the two points we connected in **a** to draw the line of intersection. The one in the xz-plane, *i.e.* $y = 0$, turns out to be slightly nicer. When $y = 0$, the equations of the planes boil down to $2x + 3z = 12$ and $6x + 3z = 24$. Simplifying the latter gives

$$6x + 3z = 24 \Longrightarrow 2x + z = 8 \Longrightarrow z = 8 - 2x\,,$$

and plugging into the former gives

$$2x + 3z = 12 \Longrightarrow 2x + 3(8 - 2x) = 12 \Longrightarrow -4x = -12 \Longrightarrow x = 3\,.$$

Plugging $x = 3$ back into $z = 8 - 2x$ yields $z = 8 - 2 \cdot 3 = 8 - 6 = 2$. Thus the point of intersection of the two given planes (which must thus be a point on the line) with the plane $y = 0$ has coordinates $(3, 0, 2)$.

Second, we find a direction vector for the line. We will do this by finding a vector perpendicular to the normal vectors of both of the given planes, which must then be parallel to both planes and hence in the direction of their line of intersection. The cross-product of the normal vectors, each of which which we can read off the equation of its plane, will be perpendicular to both:

$$\begin{aligned}\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \times \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 3 \\ 6 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 6 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ 6 & 4 \end{vmatrix}\mathbf{k} \\ &= (9 - 12)\mathbf{i} - (6 - 18)\mathbf{j} + (8 - 18)\mathbf{k} = -3\mathbf{i} + 12\mathbf{j} - 10\mathbf{k} = \begin{bmatrix} -3 \\ 12 \\ -10 \end{bmatrix}\end{aligned}$$

Thus a parametric description of the line of intersection of the two given planes is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 12 \\ -10 \end{bmatrix},$$

where t is the parameter.

The other way to find a direction vector would be to find the coordinates of another point on the line and take the vector from one point to the other as the direction vector. For those interested in this, the point of intersection of the two given planes with $z = 0$ has coordinates $\left(\frac{12}{5}, \frac{12}{5}, 0\right)$ and could, of course, also serve just as well as a base point for the parametric description of the line. ■

2. Consider the following system of linear equations.

$$\begin{array}{rcrcrcrcr} w & - & x & - & y & + & z & = & 0 \\ & & x & + & y & & & = & -1 \\ w & + & x & + & 2y & + & z & = & 1 \\ w & + & x & & & + & z & = & 2 \end{array}$$

a. Use Gaussian elimination to find all the solutions, if any, of this system. *[10]*

Solution. We'll write this system in augmented matrix form and go for it:

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 & 2 \end{array}\right] \begin{array}{c} \Longrightarrow \\ \\ R_3 - R_1 \\ R_4 - R_1 \end{array} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 2 & 3 & 0 & 1 \\ 0 & 2 & 1 & 0 & 2 \end{array}\right]$$

$$\begin{array}{c} \\ \Longrightarrow \\ R_3 - 2R_2 \\ R_4 - 2R_2 \end{array} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 0 & 4 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ \\ R_4 + R_3 \end{array} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 7 \end{array}\right]$$

At this point we can stop: since the fourth row in the final matrix has 0 in every entry except for a right-hand side of 7, the given system has no solutions. ■

b. Use your work for **a** to compute the determinant of the coefficient matrix. *[5]*

Solution. It follows from part **a** that the coefficient matrix, which is a 4×4 matrix, has rank less than 4, and hence is not invertible. Since it is not invertible, it must have determinant 0. ■

3. Find the inverse, if it exists, of $\begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$. *[10]*

Solution. We set up the required "super-augmented" matrix and use Gauss-Jordan elimination to compute the inverse, if it exists.

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2 \\ \Longrightarrow \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 7 & 2 & 1 & -2 & 0 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{array}\right] \begin{array}{c} \Longrightarrow \\ \frac{1}{7}R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1 + 2R_2 \\ \Longrightarrow \\ R_3 - 4R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{7} & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} & -\frac{6}{7} & 1 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ -7R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{7} & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right]$$

$$\begin{array}{c} R_1 + \frac{3}{7}R_3 \\ R_2 - \frac{2}{7}R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & -3 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right]$$

Thus $\begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$. ■

4. Let $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & -1 \end{bmatrix}$.

a. Use Gauss-Jordan elimination to put $\mathbf{A}$ in reduced echelon form. *[5]*

Solution.

$$\begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ \Longrightarrow \\ R_4 + R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 \end{bmatrix}$$

$$\begin{matrix} R_1 + R_2 \\ \Longrightarrow \\ R_3 + R_2 \\ R_4 + R_2 \end{matrix} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

■

b. Find bases for row($\mathbf{A}$) and col($\mathbf{A}$), the row and column spaces of $\mathbf{A}$. *[4]*

Solution. The non-zero rows of the reduced matrix in **a** are a basis for the row space, row($\mathbf{A}$):

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

The columns of the original matrix corresponding to the columns in the reduced matrix in which a leading 1 occurs in a row form a basis for the column space, col($\mathbf{A}$):

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} x \right\} \quad ■$$

c. What are the rank and nullity of $\mathbf{A}$? *[1]*

Solution. Since there are two non-zero rows in the reduced matrix in **a**, the rank of $\mathbf{A}$ is 2. The rank and nullity of $\mathbf{A}$ must add up to the number of columns of $\mathbf{A}$, so the nullity of $\mathbf{A}$ is $5 - 2 = 3$. ■

d. Find a basis for null($\mathbf{A}$), the null space of $\mathbf{A}$. *[5]*

Solution. By definition, null($\mathbf{A}$) $= \{\, \mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{0} \,\}$. Given the reduction in **a**, the equation $\mathbf{A}\mathbf{x} = \mathbf{0}$ boils down to the system of equations:

$$\begin{array}{ccccccccc} v & & & - & x & & & + & z & = & 0 \\ & & w & - & x & + & y & & & = & 0 \end{array} \quad \textit{i.e.} \quad \begin{array}{lllll} v & = x & & & - \; z \\ w & = x & - & y & \end{array}$$

Since there are two equations and five variables in the reduced system, we will require three paramaters, say r, s, and t, to write the solutions parametrically. Setting $x = r$, $y = s$, and $z = t$, means that $v = r - t$ and $w = r - s$. Writing this in vector-parametric form gives:

$$\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r-t \\ r-s \\ r \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus a basis for the null space null($\mathbf{A}$) is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \blacksquare$$

5. Find all the eigenvalues of $\mathbf{B} = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}$, and find an eigenvector for each of the eigenvalues. *[15]*

Solution. First, we find the characteristic polynomial of $\mathbf{B}$ by computing $|\mathbf{B} - \lambda\mathbf{I}|$:

$$\begin{aligned} |\mathbf{B} - \lambda\mathbf{I}| &= \left| \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 4 \\ -1 & 5-\lambda \end{vmatrix} \\ &= (-\lambda)(5-\lambda) - (-1)4 = \lambda^2 - 5\lambda + 4 \end{aligned}$$

Second, we find the eigenvalues of $\mathbf{B}$ by finding the roots of the characteristic polynomial. It would be feasible and entirely acceptable to factor this polynomial by eyeballing it or by hit and miss, but those who prefer a systematic approach can use the quadratic formula:

$$\begin{aligned} \lambda^2 - 5\lambda + 4 = 0 \Longrightarrow \lambda &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1} = \frac{5 \pm \sqrt{25 - 16}}{2} \\ &= \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2} = \frac{2}{2} \text{ or } \frac{8}{2} = 1 \text{ or } 4 \end{aligned}$$

This can be checked by noting that $(\lambda - 1)(\lambda - 4) = \lambda^2 - 5\lambda + 4$.

Third, we find an eigenvector for each of the eigenvalues $\lambda = 1$ and $\lambda = 4$ of $\mathbf{B}$. In each case, we need to find a $\mathbf{x} \neq \mathbf{0}$ such that $(\mathbf{B} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.

$\lambda = 1$: In this case, $\mathbf{B} - \lambda\mathbf{I} = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & 4 \end{bmatrix}$, so the equation $(\mathbf{B} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ boils down to $-x + 4y = 0$, *i.e.* $x = 4y$. Setting $y = 1 \neq 0$ gives $x = 4$, so $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 1$.

$\lambda = 4$: In this case, $\mathbf{B} - \lambda\mathbf{I} = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix} - 4 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -1 & 1 \end{bmatrix}$, so the equation $(\mathbf{B} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ boils down to $-x + y = 0$, *i.e.* $x = y$. Setting $y = 1 \neq 0$ gives $x = 1$, so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 4$. ■

Part II. Do any *three* of **6–11**.

6. Compute the determinant of

$$\mathbf{C} = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ -3 & 4 & 2 & -1 \end{bmatrix}$$

and use it to determine whether $\mathbf{C}$ is invertible or not. *[10]*

Solution. We will compute the determinant by expanding along the second row, because all but one of its entries are zeros. Note that because we are expanding along an even-numbered row, we must start the alternating signs with a minus.

$$|\mathbf{C}| = \begin{vmatrix} 2 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ -3 & 4 & 2 & -1 \end{vmatrix}$$

$$= -0 \cdot \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 4 & 2 & -1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ -3 & 2 & -1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 0 & 2 \\ 1 & 1 & 3 \\ -3 & 4 & -1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ -3 & 4 & 2 \end{vmatrix}$$

Now we expand the only cofactor that matters along its first row.

$$= (-1) \cdot \left[+2 \cdot \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 3 \\ -3 & -1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 2 \\ -3 & 2 \end{vmatrix} \right]$$

$$= (-1) \cdot [2 \cdot (-2 - 6) - 1 \cdot (-1 + 9) + 2 \cdot (2 + 6)]$$

$$= (-1) \cdot [-16 - 8 + 16] = (-1) \cdot [-8] = 8$$

Since $|\mathbf{C}| = 8 \neq 0$, $\mathbf{C}$ must be invertible. ■

7. Suppose $\mathbf{A}$ is a square matrix. What is $\det\left(\mathbf{A}^T\right) = \left|\mathbf{A}^T\right|$ in terms of $\det(\mathbf{A}) = |\mathbf{A}|$? Explain why! *[10]*

Solution. It turns out that $\det\left(\mathbf{A}^T\right) = \left|\mathbf{A}^T\right| = |\mathbf{A}| = \det(\mathbf{A})$. (This is Theorem 4.10 in the text.) The reason is that the rows (respectively, columns) of $\mathbf{A}$ are the columns (respectively, rows) of $\mathbf{A}^T$. Thus, if one computes $|\mathbf{A}|$ (and any relevant minors of $\mathbf{A}$) by expanding along a row (respectively, column) one encounters the same entries in the same order as one does by computing $\left|\mathbf{A}^T\right|$ (and any relevant minors of $\mathbf{A}^T$) by expanding $\mathbf{A}^T$ along the corresponding columns (respectively, rows). ■

8. Find all 2×2 matrices $\mathbf{X}$ satisfying the matrix equation $\mathbf{X}^2 + \mathbf{X} - 2\mathbf{I}_2 = \mathbf{O}_2$. *[10]*

Solution. An obvious thing to try is to factor the given matrix equation. Note that the polynomial $x^2 - x + 2 = (x+2)(x-1)$, and the corresponding factorization works for the given matrix polynomial:

$$\mathbf{X}^2 + \mathbf{X} - 2\mathbf{I}_2 = \left(\mathbf{X} + 2\mathbf{I}_2\right)\left(\mathbf{X} - \mathbf{I}_2\right)$$

It is very tempting to assume that because $(x+2)(x-1) = 0$ implies that $x + 2 = 0$ or $x - 1 = 0$, and hence that $x = -2$ or $x = 1$, that the corresponding argument works for the matrix equation:

$$\begin{aligned}\left(\mathbf{X} + 2\mathbf{I}_2\right)\left(\mathbf{X} - \mathbf{I}_2\right) = \mathbf{0}_2 \quad &\Longrightarrow \quad \mathbf{X} + 2\mathbf{I}_2 = \mathbf{0}_2 \quad \text{or} \quad \mathbf{X} - \mathbf{I}_2 = \mathbf{0}_2 \\ &\Longrightarrow \quad \mathbf{X} = -2\mathbf{I}_2 \quad \text{or} \quad \mathbf{X} = \mathbf{I}_2\end{aligned}$$

Unfortunately, this argument breaks down at the first step because it is perfectly possible to have matrices $\mathbf{A} \neq \mathbf{0}_2$ and $\mathbf{B} \neq \mathbf{0}_2$ with $\mathbf{AB} = \mathbf{0}_2$. Mind you, the conclusion is partially correct, in that $\mathbf{X} = -2\mathbf{I}_2$ and $\mathbf{X} = \mathbf{I}_2$ are both solutions of the given matrix equation, but they are not the only possible solutions. For example, $\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ is also a solution of $\mathbf{X}^2 + \mathbf{X} - 2\mathbf{I}_2 = \mathbf{O}_2$.

The foregoing, or even most of it, would have been enough for full credit on this question, but to answer it fully requires one to get down and dirty with the possible entries of a 2×2 matrix $\mathbf{X}$ such that $\mathbf{X}^2 + \mathbf{X} - 2\mathbf{I}_2 = \mathbf{O}_2$. Suppose then that $\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a solution to this matrix equation. Then

$$\begin{aligned}\mathbf{X}^2 + \mathbf{X} - 2\mathbf{I}_2 &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 + \begin{bmatrix} a & b \\ c & d \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} a^2 + bc + a - 2 & ab + bd + b \\ ac + cd + c & bc + d^2 + d - 2 \end{bmatrix},\end{aligned}$$

so we need to find all the solutions of

$$\begin{bmatrix} a^2 + bc + a - 2 & ab + bd + b \\ ac + cd + c & bc + d^2 + d - 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

i.e. all the solutions to the (non-linear) system of equations

$$\begin{aligned} a^2 + bc + a - 2 &= 0 \qquad & ab + bd + b &= 0 \\ ac + cd + c &= 0 \qquad & bc + d^2 + d - 2 &= 0\,. \end{aligned}$$

We first rewrite and rearrange these equations to make them a little easier to work with:

$$\begin{gathered} (a+2)(a-1) = a^2 + a - 2 = -bc \\ (d+2)(d-1) = d^2 + d - 2 = -bc \\ b(a+d+1) = 0 \\ c(a+d+1) = 0 \end{gathered}$$

Note that it follows from the last two equations that either $b = 0$ and $c = 0$, or $a+d+1 = 0$. We will consider each of these possibilities separately.

I. $b = 0$ and $c = 0$: In this case $-bc = 0$, so $(a+2)(a-1) = 0$ and $(d+2)(d-1) = 0$, from which it follows that $a = -2$ or $a = 1$ and that $d = -2$ or $d = 1$. This gives the following four solutions to the given matrix equation:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \qquad \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

Note that the middle two of these solutions make $a + d + 1 = 0$ true as well.

II. $a + d + 1 = 0$: We need to consider three sub-cases. (The fourth possible subcase, where $b = 0$ and $c = 0$, has already been dealt with above.)

i. $b = 0$ and $c \neq 0$: In this case $-bc = 0$, so, as above, we get that $a = -2$ or $a = 1$ and that $d = -2$ or $d = 1$. Since we must also satisfy $a + d + 1 = 0$, this boils down to either $a = -2$ and $d = 1$ or $a = 1$ and $d = -2$. However, we have no further restrictions on c, so we get the following (infinite families of) solutions to the given matrix equation:

$$\begin{bmatrix} -2 & 0 \\ c & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ c & -2 \end{bmatrix}$$

ii. $b \neq 0$ and $c = 0$: In this case $-bc = 0$, so, as above, we get that $a = -2$ or $a = 1$ and that $d = -2$ or $d = 1$. Since we must also satisfy $a + d + 1 = 0$, this boils down to either $a = -2$ and $d = 1$ or $a = 1$ and $d = -2$. However, we have no further restrictions on b, so we get the following (infinite families of) solutions to the given matrix equation:

$$\begin{bmatrix} -2 & b \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & b \\ 0 & -2 \end{bmatrix}$$

iii. $b \neq 0$ and $c \neq 0$: In this case $-bc \neq 0$, so we have to work harder to solve for a and d. To solve $a^2 + a - 2 + bc = 0$ for a and $d^2 + d - 2 + bc = 0$ for d, we use the quadratic formula, which yields

$$\begin{aligned} a &= \frac{-1 \pm \sqrt{1^2 - 4(-2+bc)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{9 - 4bc}}{2} = -\frac{1}{2} \pm \sqrt{\frac{9}{4} - bc} \quad \text{and} \\ d &= \frac{-1 \pm \sqrt{1^2 - 4(-2+bc)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{9 - 4bc}}{2} = -\frac{1}{2} \pm \sqrt{\frac{9}{4} - bc}\,. \end{aligned}$$

That is, we have two possibilities for each of a and d, $-\frac{1}{2}+\sqrt{\frac{9}{4}-bc}$ and $-\frac{1}{2}-\sqrt{\frac{9}{4}-bc}$.

If one of a or d is $-\frac{1}{2}+\sqrt{\frac{9}{4}-bc}$ and the other is $-\frac{1}{2}-\sqrt{\frac{9}{4}-bc}$, then $a+d+1=0$ and there is no further restriction on bc except that we need $\frac{9}{4}-bc \geq 0$, *i.e.* $bc \leq \frac{9}{4}$, for the square root to make sense. In this situation we get the following infinite families of solutions to the matrix equation,

$$\begin{bmatrix} -\frac{1}{2}+\sqrt{\frac{9}{4}-bc} & b \\ c & -\frac{1}{2}-\sqrt{\frac{9}{4}-bc} \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{2}-\sqrt{\frac{9}{4}-bc} & b \\ c & -\frac{1}{2}+\sqrt{\frac{9}{4}-bc} \end{bmatrix},$$

where b and c need only satisfy $0 \neq bc \leq \frac{9}{4}$.

If $a=d=-\frac{1}{2}+\sqrt{\frac{9}{4}-bc}$, then $a+d+1=0$ gives $2\sqrt{\frac{9}{4}-bc}=0$, from which it follows that $bc=\frac{9}{4}$. In this situation we get the following infinite family of solutions to the matrix equation,

$$\begin{bmatrix} -\frac{1}{2}+\sqrt{\frac{9}{4}-bc} & b \\ \frac{9}{4b} & -\frac{1}{2}+\sqrt{\frac{9}{4}-bc} \end{bmatrix},$$

where b need only satisfy $b \neq 0$.

If $a=d=-\frac{1}{2}-\sqrt{\frac{9}{4}-bc}$, then $a+d+1=0$ gives $-2\sqrt{\frac{9}{4}-bc}=0$, from which it follows that $bc=\frac{9}{4}$. In this situation we get the following infinite family of solutions to the matrix equation,

$$\begin{bmatrix} -\frac{1}{2}-\sqrt{\frac{9}{4}-bc} & b \\ \frac{9}{4b} & -\frac{1}{2}-\sqrt{\frac{9}{4}-bc} \end{bmatrix},$$

where b need only satisfy $b \neq 0$.

The truly mathochistic can amuse themselves here by considering the degree to which these various sets of solutions overlap each other, by way of simplifying their total description ... ■

9. Suppose T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}4\\4\\0\end{bmatrix},\quad T\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)=\begin{bmatrix}4\\0\\4\end{bmatrix},\quad \text{and}\quad T\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)=\begin{bmatrix}0\\4\\4\end{bmatrix}.$$

What is the matrix $\mathbf{A}_T$ associated to this linear transformation? (This matrix is called the *standard matrix of* T and denoted by $[T]$ in the text.) *[10]*

Solution. The columns of $A_T = [T]$ are $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$, and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$, so our first step will be to write $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ in terms of $\begin{bmatrix}-1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\-1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\1\\-1\end{bmatrix}$. Fortunately, this is pretty easy to do by inspection. Since $\begin{bmatrix}-1\\1\\1\end{bmatrix}+\begin{bmatrix}1\\-1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\2\end{bmatrix}$, $\begin{bmatrix}-1\\1\\1\end{bmatrix}+\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}0\\2\\0\end{bmatrix}$, and $\begin{bmatrix}1\\-1\\1\end{bmatrix}+\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}2\\0\\0\end{bmatrix}$, we have $\begin{bmatrix}1\\0\\0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\1\\-1\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1\\1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\1\\-1\end{bmatrix}$, and $\begin{bmatrix}0\\0\\1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1\\1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}$.

Since T is a linear transformation, it now follows that

$$\begin{aligned}
T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) &= T\left(\frac{1}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)\\
&= \frac{1}{2}T\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)+\frac{1}{2}T\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)\\
&= \frac{1}{2}\begin{bmatrix}4\\0\\4\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0\\4\\4\end{bmatrix}=\begin{bmatrix}2\\2\\4\end{bmatrix},
\end{aligned}$$

and, similarly, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\4\\2\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}4\\2\\2\end{bmatrix}$.

Thus $A_T = [T] = \begin{bmatrix}2&2&4\\2&4&2\\4&2&2\end{bmatrix}$. ■

10. Find a basis for the subspace $S = \text{Span}\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$. *[10]*

Solution. We assemble the vectors of the given spanning set into the rows of a matrix and use Gaussian elimination to reduce the matrix:

$$\begin{bmatrix} 3 & -1 & 1 & -1 \\ -1 & 3 & 1 & -1 \\ -1 & -1 & 1 & 3 \\ 1 & 1 & 3 & 1 \end{bmatrix} \overset{R_1 \leftrightarrow R_4}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 3 & 1 \\ -1 & 3 & 1 & -1 \\ -1 & -1 & 1 & 3 \\ 3 & -1 & 1 & -1 \end{bmatrix}$$

$$\begin{array}{c} \Longrightarrow \\ R_2 + R_1 \\ R_3 + R_1 \\ R_4 - 3R_1 \end{array} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & -4 & -8 & -4 \end{bmatrix} \begin{array}{c} \\ \Longrightarrow \\ \\ R_4 + 3R_2 \end{array} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & -4 & -4 \end{bmatrix}$$

$$\begin{array}{c} \\ \frac{1}{4}R_2 \\ \Longrightarrow \\ R_4 + 3R_3 \end{array} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \\ \Longrightarrow \\ \frac{1}{4}R_3 \\ \\ \end{array} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can read off the required basis for S from the non-zero rows of the reduced matrix:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad \blacksquare$$

11. Find the distance from the point $(2, 0, 1)$ in $\mathbb{R}^3$ to the plane given by the equation $x - y - z = -1$. *[10]*

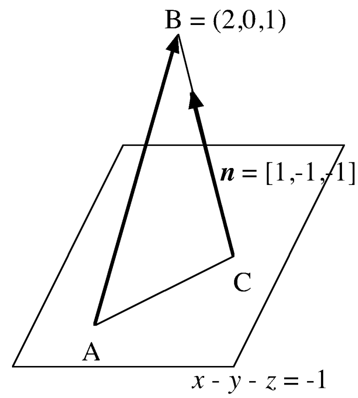
Solution. The quickest way to do this is to plug everything into formula (4) from Section 1.3 of the text: the distance between the point $B = (x_0, y_0, z_0)$ and the plane $\mathcal{P}$ whose equation is $ax + by + cz = d$ is given by $\mathrm{d}(B, \mathcal{P}) = \frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$.

Thus the distance between the given point and plane is:

$$\frac{|1 \cdot 2 - 1 \cdot 0 - 1 \cdot 1 - (-1)|}{\sqrt{1^2 + (-1)^2 + (-1)^2}} = \frac{2}{\sqrt{3}}$$

To be fair, this is the kind of formula not everyone will remember, or even write down on their aid sheet, so here is a more basic approach:

The shortest path from the given point $B = (2, 0, 1)$ to a plane will run perpendicular to the plane from B to a point C on the plane, so it will be parallel to the normal vector $\mathbf{n} = [1, -1, -1]$ to the plane. If we pick some point A on the plane, then the distance from B to C, $|BC|$, will the equal to the length of the component of the vector AB which is



parallel to the normal vector $\mathbf{n}$. That is, it will be equal to the length of the projection of the vector AB onto $\mathbf{n}$, $\mathrm{proj}_{\mathbf{n}}(AB)$.

The above outlines what we'll do:

First, we find a point A on the plane $x - y - z = -1$. To do this, we need only set $y = z = 0$ and solve for x, which will give us $A = (-1, 0, 0)$.

Second, we compute the vector AB by taking the difference of the coordinates of C and A: $AC = [2 - (-1), 0 - 0, 1 - 0] = [3, 0, 1]$.

Third, we compute $\mathrm{proj}_{\mathbf{n}}(AB)$:

$$\begin{aligned}\mathrm{proj}_{\mathbf{n}}(AB) &= \left(\frac{\mathbf{n}\cdot AB}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \left(\frac{[1,-1,-1]\cdot[3,0,1]}{[1,-1,-1]\cdot[1,-1,-1]}\right)[1,-1,-1] \\ &= \left(\frac{1\cdot 3 + 0\cdot(-1) + 1\cdot(-1)}{1^2+(-1)^2+(-1)^2}\right)[1,-1,-1] = \frac{2}{3}[1,-1,-1] \\ &= \left[\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right]\end{aligned}$$

Finally, we compute the length of $\mathrm{proj}_{\mathbf{n}}(AB)$:

$$\begin{aligned}\|\mathrm{proj}_{\mathbf{n}}(AB)\| &= \left\|\left[\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right]\right\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} \\ &= \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{4}{9}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}\end{aligned}$$

Hence the distance from the point $(2, 0, 1)$ to the plane $x - y - z = -1$ is $\frac{2}{\sqrt{3}}$. ■

[Total = 95]

Part Null. Bonus!

0^{0^0}**.** Write an original little poem about linear algebra or mathematics in general. *[2]*

Solution. You're on your own on this one! ■